Graphs and polyhedra

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GRAPHS AND POLYHEDRA

BINARY SPACES AND CUTTING PLANES

A.M.H. Gerards

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Proefschrift

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aan Yvonne, Mark en Rob
en mijn ouders

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SUMMARY OF RESULTS

This thesis is partly based on (parts of) the following articles and reports: Cook, Gerards, Schrijver and Tardos [1986], Gerards [1985, 1986] and Gerards and Schrijver [1986], and on a forthcoming report by Gerards, Lovász, Schrijver, Seymour, Shih and Truemper. Below we give a summary of the main results in this thesis.

It should be noted that in this summary we sometimes use a formulation different from the text in this monograph. As Chapter 1 is only an introduction to the four fields in mathematics (Computational Complexity, Polyhedral Theory, Graphs and Signed Graphs, and Binary matroids = Binary Spaces) relevant for this thesis, we restrict ourselves to the three other chapters.

CUTTING PLANES (CHAPTER 2)

Consider a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqq b\}$ (A rational). We are interested in describing P_I := convex hull $(P \cap \mathbb{Z}^n)$ by a system of inequalities. A *cutting plane* for P is an inequality

$$c^T x \leqq \lfloor \delta \rfloor$$
with $c \in \mathbb{Z}^n$
and $\delta \geqq \max\{c^T x \mid x \in P\}$.

The set of all vectors satisfying all cutting planes for P is denoted by P'. We define $P^{(0)} := P$, and $P^{(i+1)} := (P^{(i)})'$ $(i=0,1,\ldots)$, and say that P has *Chvátal rank* t if t is the smallest integer such that $P^{(t)}$ = convex hull $(P \cap \mathbb{Z}^n)$. Chvátal [1973] and Schrijver [1980] proved that each polyhedron has such a (finite) Chvátal rank. In fact it can be bounded by an integer depending on A only (so independent of the right hand side b). A short proof of this is given in Section 2.2.

The central result however in Chapter 2, and in this thesis is

Theorem 2.3.3 (Gerards and Schrijver [1986])
Let $A \in \mathbb{Z}^{m \times n}$ *such that* $\sum_{j=1}^{n} |A_{ij}| \leqq 2$ *for each* $i=1,\ldots,m$. *Then the following are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid d_1 \leqq x \leqq d_2,\ b_1 \leqq Ax \leqq b_2\}$ *has Chvátal rank at most* 1 *for all* $d_1, d_2 \in \mathbb{Z}^n$ *and* $b_1, b_2 \in \mathbb{Z}^m$;

(ii) The signed graph underlying A contains no odd-K_4. □

Here a *signed graph* is an undirected graph with a partition of the edges into *odd* and *even* edges. If $A \in \mathbb{Z}^{m \times n}$ satisfies $\sum_{j=1}^{n} |A_{ij}| \leqq 2$ for each $i=1,\ldots,m$, then the *signed graph underlying* A is constructed as follows. First construct the undirected graph with as nodes the columns of A. For each row of A with two non-zero entries we have an edge joining these two columns in which these two non-zero entries occur. We call an edge *even* if the corresponding row sum is 0, if not we call the edge *odd*. An *odd-*K_4 is a signed homeomorph of K_4 (the complete graph on 4 nodes) such that each circuit coming from a triangle in K_4 is an *odd circuit* (i.e. a circuit with an odd number of odd edges).

Theorem 2.3.3 shows that recognizing whether or not a matrix $A \in \mathbb{Z}^{m \times n}$ satisfying $\sum_{j=1}^{n} |A_{ij}| \leqq 2$ for each $i=1,\ldots,m$, satisfies Theorem 2.3.3 amounts to recognizing graphs with no odd K_4. This is one of the reasons for further investigation of such signed graphs in Chapter 3.

SIGNED GRAPHS WITH NO ODD-K_4 (CHAPTER 3)

Examples of such signed graphs are:

- Signed graphs in which all odd circuits have a node in common;
- Signed graphs which can be embedded in the plane such that at most two faces are bounded by an odd circuit.

Essentially, these are the only examples. Each signed graph with no odd-K_4 can be "decomposed" into these examples and two small special signed graphs (Theorem 3.2.4). This result implies a polynomial-time algorithm for recognizing signed graphs with no odd-K_4. The proof is based on decomposition results for binary matroids (= binary spaces) due to Seymour [1980] and to Truemper and Tseng [1986], applied to a binary matroid associated with a signed graph.

Beside the decomposition result mentioned we prove:

Theorem 3.3.1
A signed graph G has no odd-K_4 and no, so called, odd-K_3^2 (cf. Section 3.1) if and only if we can replace the odd edges by directed edges, such that going along any circuit the number of forwardly directed edges and the number of backwardly directed edges differ by at most 1. □

This is derived from Tutte's characterization of regular matroids (= binary spaces representable in euclidean space = totally unimodular matrices, cf. Section 4.1, which contains a short proof of Tutte's result).

Theorem 3.3.1 has several interesting implications. In Section 3.5 we use it to obtain a short proof (due to A. Schrijver) of the following extension of a result of Albertson, Catlin and Gibbons [1985].

Theorem 3.5.1 (Gerards [1985])
Let G be an undirected non-bipartite graph such that there is no odd-K_4 and no odd-K_3^2 (considering all edges odd). Then there exists an map φ from the nodes of G to the nodes on the shortest odd circuit of G such that if uv is an edge, then $\varphi(u)\varphi(v)$ is an edge. □

Theorem 3.3.1 also plays an important role in proving the following extension of König's min-max relation for stable sets and edge-covers in bipartite graphs (cf. (3.6.1), König [1931, 1933]).

Theorem 3.6.3 (Gerards [1986])
Let G be an undirected graph, without isolated nodes, such that there is no odd-K_4 (considering all edges odd). Then the

maximum cardinality of a stable set in G

is equal to the

minimum cost of a collection of edges and odd circuits covering the nodes of G.

(Here an edge costs 1 *and a circuit of length* $2k+1$ *costs* k*.)* □

A weighted version of this result, and of a similar result for node-covers also holds (cf. Theorems 3.6.3 and 3.6.8).

T-JOINS (CHAPTER 4)

In Section 4.2 we prove the following extension of a result of Seymour [1981].

Theorem 4.2.2

Let G *be a connected undirected graph such that (considering all edges odd) there is no odd-*K_4 *and no, so-called, odd-prism (cf. Section 4.2, Figure 4.2). Then for each even set* T *of nodes the*

minimum cardinality of a T*-join in* G

is equal to the

maximum number of pairwise disjoint T*-cuts in* G. □

Here a set F of edges is a T-*join* if a node u of G meets an odd number of edges in F if and only if $u \in T$. A T-*cut* is a set of edges of the form $\{uv \mid u \in U, v \notin U\}$ where U is a set of nodes with $|U \cap T|$ odd.

In Section 4.3 until 4.6 we derive results for T-joins which are dual to the results derived in Sections 3.1 and 3.4.

CHAPTER 1. PRELIMINARIES

This chapter contains four preliminary sections, viz. on: Algorithms and Complexity, Polyhedral Theory, Graphs and Signed Graphs, and Binary Matroids = Binary Spaces. This chapter intends to be an introduction, rather than an extensive treatment. Therefore proofs are omitted except in Section 1.4.

1.1. ALGORITHMS AND COMPLEXITY

A main objective of studying objects like those studied in this monograph, is finding efficient algorithms. In fact, most of the results imply efficient algorithms. (However, detailed descriptions of these algorithms will not be explicitly given.) We here give a brief and intuitive introduction to algorithms and complexity. For a detailed treatment we refer to Aho, Hopcroft and Ullman [1974] and Garey and Johnson [1979].

We consider an algorithm as a recipe, i.e. as a list of instructions, such that if we apply this recipe to an "input" we get after a finite number of applications of the instructions, an "output". The *running time* of an algorithm is the number of "elementary" steps it takes, as a function of the size of an input. This definition of running time depends on what we consider as an elementary step. Often a single bit operation on a computer or a move of the head of a Turing Machine is considered as an *elementary step*.

The *size* of the input is the number of digits needed to encode the input. For example, if we encode a natural number n in binary notation its (input) size is about ${}^{2}\log(n)$; the size of a rational number is the size of its denominator plus the size of its numerator. Of course, the input of an algorithm need not be a number; for instance, it can be a graph. As the input size of a graph we take the number of nodes plus the number of edges.

If the running time of an algorithm is bounded from above by a polynomial in the input size, we call the algorithm a *polynomial-time algorithm*. The search for polynomial-time algorithms has led to a classification of problems into easy and (possibly) hard problems. To explain this we restrict ourselves to a specific type of problems: so-called decision problems. A *decision problem* is a problem which allows for each input a 'yes' or 'no' answer. Let us give some examples.

(CONNECTED GRAPH) Given an undirected graph G, is G connected?

(LI) Given a system of linear inequalities $Ax \leqq b$, with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^n$, is there an $\hat{x} \in \mathbb{Q}^n$ such that $A\hat{x} \leqq b$?

(HAMILTONIAN CIRCUIT) Given an undirected graph G has it a Hamiltonian circuit (i.e., is there a permutation $v_1, \ldots, v_n$ of the nodes of G such that v_1v_2, $v_2v_3, \ldots, v_{n-1}v_n$ and v_nv_1 all are edges of G)?

To distinguish between a specific question like "has $x_1 - 2x_2 \leqq 3$, $x_1 \leqq 0$ a solution?" and the collection of all questions defined by LI, we call "has $x_1 - 2x_2 \leqq 3$, $x_1 \leqq 0$ a solution?" an *instance* of the problem LI.

PROBLEMS

The class of all decision problems which can be solved by a polynomial-time algorithm is denoted by $\mathcal{P}$. It is easy to see that CONNECTED GRAPH $\in \mathcal{P}$.

WELL-CHARACTERIZED PROBLEMS

Khachiyan [1979] showed that LI$\in\mathcal{P}$. However even before that it was already known that LI is reasonable to some extent. To explain what we mean by that we consider Farkas Lemma:

(1.1.1) Farkas [1894]: Let $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Then exactly one of the following holds:

- (i) There exists an $x \in \mathbb{Q}^n$ with $Ax \leqq b$;
- (ii) There exists an $y \in \mathbb{Z}^m$ with $y^T A = 0$, $y \geqq 0$ and $y^T b < 0$.

It follows from (1.1.1) that we can attach to each instance $Ax \leqq b$ of LI a guarantee for the status (having a solution or not) of the system $Ax \leqq b$. Namely if $Ax \leqq b$ has a solution, then a guarantee of that fact is a vector

$$x \in \mathbb{Q}^n \text{ with } Ax \leqq b.$$

If $Ax \leqq b$ has no solution a guarantee of that fact is a vector

$$y \in \mathbb{Q}^n \text{ with } y^T A = 0,\ y \geqq 0 \text{ and } y^T b < 0.$$

This means, that for each instance of LI we can provide, beside the answer ("yes" or "no"), a "proof" of the correctness of the answer. We call such a "proof" a *certain certificate* for the instance of LI. In general, a *certain certificate* for an instance of a decision problem is a list of symbols reflecting a proof of the correctness of the answer ('yes' or 'no') of the instance. The *length* of a certain certificate is the number of elementary steps to read and check the certain certificate.

A decision problem P is called *well-characterized* if each instance of P has a *polynomial-length* certain certificate; this means that the length of that certain certificate is a polynomial in the size of the instance. The certain certificates for instances of LI given above can be taken of polynomial length. Hence LI is well-characterized. It should be noted that, in defining well-characterized problems, we did not require the existence of a polynomial-time algorithm to find a certain certificate for any instance of the problem. If such algorithm exists for some problem then clearly that problem is in $\mathcal{P}$. Although, as mentioned before, LI$\in\mathcal{P}$, it is open whether or not all well-characterized problems are in $\mathcal{P}$. In particular, it is open whether or not the decision problem "given a natural number p, is it prime?" (which is well-characterized, (Pratt [1975]), is in $\mathcal{P}$.

Problems in $\mathcal{P}$ are well-characterized. Indeed, suppose we have a polynomial-time algorithm for a problem P. Then a certain certificate for the answer to an instance is the instance itself. This certain-certificate has polynomial-length as it can be checked by the polynomial-time algorithm for the problem.

The fact that a problem P is well-characterized is often established by a so-called good characterization. To explain this notion we turn back to LI and consider the following equivalence:

(1.1.2) Let $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^n$. Then the following are equivalent:

(i) There exists an $x \in \mathbb{Q}^n$ with $Ax \leqq b$.
(ii) The matrix $[A|-A|I]$ has a non-singular submatrix $B \in \mathbb{Z}^{m\times m}$, with $B^{-1}b \geqq 0$.

So (1.1.2) provides a characterization for a system of linear inequalities to have a solution. Mathematically there is nothing wrong with this characterization, but from the point of view of computational complexity it has a drawback. The reason is that (1.1.2) only tells us (two ways) how to show easily that a system of linear inequalities has a solution. How to show that a given system of linear inequalities, is not so obvious from (1.1.2). Farkas Lemma (1.1.1) does not have this draw-back. For that reason we call Farkas Lemma a good characterization for LI.

In general, if P is a decision problem we call a characterization *good* if it establishes polynomial-length certain certificates for the instances of P. The term "good characterization" has been introduced by Edmonds [1965b].

$\mathcal{NP}$ AND co-$\mathcal{NP}$

By $\mathcal{NP}$ one denotes the class of decision problems for which there exists a polynomial-length certain certificate for each instance having a 'yes' answer. ($\mathcal{NP}$ stands for $\mathcal{P}$olynomially solvable by a $\mathcal{N}$ondeterministic Turing machine, cf. Garey and Johnson [1979].) Of course, well-characterized problems are in $\mathcal{NP}$. But there may be problems in $\mathcal{NP}$, for which there exists no good characterization. Consider HAMILTONIAN CIRCUIT. If a graph is hamiltonian, then any hamiltonian circuit may serve as a certain certificate. Hence HAMILTONIAN CIRCUIT $\in \mathcal{NP}$. On the other hand, no polynomial-length certain certificate for the fact that a graph has no hamiltonian circuit is known. In other words, it is open whether HAMILTONIAN CIRCUIT is well-characterized.

By co-$\mathcal{NP}$ one denotes the class of all decision problems for which there exists a polynomial-length certain certificate for each instance having a 'no' answer. Obviously, $\mathcal{NP} \cap$ co-$\mathcal{NP}$ is exactly the class of well-characterized problems.

$\mathcal{NP}$-COMPLETE PROBLEMS

We call a problem P $\mathcal{NP}$-complete if P$\in\mathcal{NP}$ and for each problem P'$\in\mathcal{NP}$ there exists a polynomial-time algorithm which transforms each instance I' of P' to an instance I of P such that the answer to I' is the same as the answer to I. Cook [1971] showed that $\mathcal{NP}$-complete problems exist. In particular, he showed that SATISFIABILITY (cf. Garey and Johnson [1979]) is $\mathcal{NP}$-complete. Our example, HAMILTONIAN CIRCUIT is $\mathcal{NP}$-complete too, and there are many others (cf. Karp [1972], Garey and Johnson [1979], and the periodically published list "$\mathcal{NP}$-complete problems: an ongoing column" by D. Johnson in the Journal of Algorithms). No polynomial-time algorithm is found for any $\mathcal{NP}$-complete problem. Note that if there exists a polynomial algorithm for one $\mathcal{NP}$-complete problem, then any problem in $\mathcal{NP}$ is polynomially solvable. So, in that case $\mathcal{NP} = \mathcal{P}$. In fact $\mathcal{NP}$-complete problems are notorious for their intractability in practice. This leads to the conjecture that $\mathcal{P} \neq \mathcal{NP}$.

Finally, let P be a problem (not necessarily a decision problem). Problem P is called $\mathcal{NP}$-*hard* if the existence of a polynomial-time algorithm for P would imply $\mathcal{NP} = \mathcal{P}$.

OPTIMIZATION PROBLEMS, MIN-MAX RELATIONS AS GOOD CHARACTERIZATIONS

Suppose we have a problem with instances:

(1.1.3) Given set X_i and function $f_i\colon X_i \rightarrow \mathbb{R}$, find an $\hat{x}\in X_i$ such that $f_i(\hat{x}) = \min\{f_i(x) \mid x\in X_i\}$ or decide that no such $\hat{x}$ exists.

(with i element of some index set I).

We call such a problem an *optimization problem*, or more specifically, a *minimization problem*. (Similarly we have *maximization problems*.) The set X_i is called the *solution set* of (1.1.3). Any member of X_i is called a *feasible solution* of (1.1.3). If $\hat{x} \in X_i$ attains the minimum in (1.1.3), we call $\hat{x}$ an *optimal solution* of (1.1.3). The value $\min\{f_i(x) | x \in X_i\}$ is called the *optimum value* of (1.1.3).

A *min-max relation* for (1.1.3) is a theorem like:

(1.1.4) $\min\{f_i(x) | x \in X_i\} = \max\{g_i(y) | y \in Y_i\} \qquad (i \in I).$

Often a min-max relation is a good characterization for the decision problem:

(1.1.5) Given $\hat{x} \in X_i$, is $\hat{x}$ an optimal solution of (1.1.3)?

Indeed, a certain certificate for $\hat{x}$ being optimal is a $\hat{y} \in Y_i$, with $f_i(\hat{x}) = g_i(\hat{y})$. A certain certificate for $\hat{x} \in X_i$ being non-optimal is an $\tilde{x} \in X_i$ with $f_i(\tilde{x}) < f_i(\hat{x})$. Depending how the optimization problems in (1.1.4) are formulated, one can obtain polynomial-length versions of these certain certificates. (Indeed $\tilde{x}$ and $\hat{y}$ should have polynomial size. Moreover membership of $\tilde{x}$ in X_i and $\hat{y} \in Y_i$ should be verifiable in polynomial-time. Finally evaluating $f_i(\hat{x})$, $f_i(\tilde{x})$ and $g_i(\hat{y})$ should take only polynomial-time.)

We call optimization problem (1.1.4) *well-characterized* if decision problem (1.1.5) is well-characterized. So if with an optimization problem there is a min-max relation, then (under some extra conditions on the formulation of the problem and the min-max relation, see above) the optimization problem is well-characterized.

An example is the linear programming problem and the linear programming duality theorem (von Neumann [1947], Gale, Kuhn and Tucker [1961], cf. Theorem 1.2.6 of this monograph). Also many combinatorial optimization problems have a min-max relation. (e.g. Theorem 3.4.1 (with special cases: Theorem 3.4.2 and Theorem 4.6.1), König's Theorem ((3.6.1)) (with extensions: Theorem (3.6.3) and Theorem (3.6.8)), and Theorem 4.1.1 (with extension: Theorem 4.2.2).

1.2. POLYHEDRAL THEORY

This section is devoted to polyhedra, optimizing a linear functional over a polyhedron (linear programming), and integral polyhedra (which arise often in combinatorial optimization problems). The introduction we give here is very condensed. Almost all proofs are omitted, and we only mention the results relevant for this monograph. For a comprehensive study we recommend Schrijver [1986].

First we make some notational conventions on numbers, vectors, matrices, etc..

NUMBERS, VECTORS AND MATRICES

We denote the sets of reals, of rationals, and of integers by $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ respectively. The set of non-negative reals is denoted by $\mathbb{R}_+$. Similarly we write $\mathbb{Q}_+$ and $\mathbb{Z}_+$ (=$\mathbb{N}$). If $\alpha\in\mathbb{R}$ then $\lfloor\alpha\rfloor$ denotes the largest integer not greater than α. Similarly $\lceil\alpha\rceil$ denotes the smallest integer not smaller than α.

Vectors are always considered as column vectors. The set of n-dimensional vectors with entries in a set S is denoted by S^n. For example we write $\mathbb{R}^n$, $\mathbb{R}^n_+$, $\{0,1\}^n$ etc. The set of m×n-matrices (m rows, n columns) with real variables is denoted by $\mathbb{R}^{m\times n}$. If $A\in\mathbb{R}^{m\times n}$, then $r(A)$ denotes the rank of A. A^T denotes the transpose of the matrix A. Row vectors are typically written as x^T.

We write $x \geqq 0$ if $x\in\mathbb{R}^n_+$. We write $x \geqq y$ if $x-y \geqq 0$. A system of m inequalities in n variables is typically written as $Ax \leqq b$ (with $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^m$). If $x\in\mathbb{R}^n$, then $\lfloor x\rfloor := (\lfloor x_1\rfloor,\ldots,\lfloor x_n\rfloor)^T$; similarly $\lceil x\rceil := (\lceil x_1\rceil,\ldots,\lceil x_n\rceil)^T$.

In combinatorial optimization we often use vectors indexed over some finite set S. Then we typically do not assume some numbering $1,\ldots,|S|$ of the entries of the vectors. So we write $\mathbb{R}^S$ rather then $\mathbb{R}^{|S|}$. If $x\in\mathbb{R}^S$, and $s\in S$ then x_s denotes the entry of x indexed by s. Similarly we write $A\in\mathbb{R}^{S\times T}$, to denote a matrix where the rows are indexed by a finite set S and the columns by a finite set T. We use A_{st} for the entry in the row of A indexed by $s\in S$, and the column of A indexed by $t\in T$. If S is a finite set and $T \subset S$ then the *characteristic vector* χ_T of T is the vector in $\{0,1\}^S$ with $\chi_{T,e} = 1$ if and only if $e \in T$.

POLYHEDRA AND POLYTOPES

A *halfspace* in $\mathbb{R}^n$, is a set of the form $\{x \in \mathbb{R}^n | a^T x \leqq \beta\}$ where $a \in \mathbb{R}^n$, $a \neq 0$, and $\beta \in \mathbb{R}$. A *polyhedron* in $\mathbb{R}^n$ is the intersection of finitely many halfspaces. So $P \subset \mathbb{R}^n$ is a polyhedron in $\mathbb{R}^n$ if and only if there exists a matrix $A \in \mathbb{R}^{m \times n}$, and a vector $b \in \mathbb{R}^m$ ($m \in \mathbb{N}$) such that $P = \{x \in \mathbb{R}^n | Ax \leqq b\}$. We call an inequality *valid* for $P \subset \mathbb{R}^n$, if $\hat{x} \in P$ implies $a^T \hat{x} \leqq \beta$. A halfspace H is called *rational* if $H = \{x \in \mathbb{R}^n | a^T x \leqq \beta\}$ with $a \in \mathbb{Q}^n$, $\beta \in \mathbb{Q}$. The intersection of a finite number of rational halfspaces is called a *rational polyhedron*.

A *polytope* P in $\mathbb{R}^n$ is the convex hull of finitely many vectors in $\mathbb{R}^n$. So $P \subset \mathbb{R}^n$ is a polytope if there exists a finite number of vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ such that

$$P = \text{conv}\{x_1, \ldots, x_m\} := \{\sum_{i=1}^m \lambda_i x_i | \lambda_i \geqq 0 \ (i=1,\ldots,m), \ \sum_{i=1}^n \lambda_i = 1\}.$$

If $x_1, \ldots, x_m$ are in $\mathbb{Q}^n$ we call P a *rational polytope*. Polytopes obviously are bounded sets, whereas polyhedra can be unbounded ($\mathbb{R}^n$ itself is a polyhedron). However the two concepts are very close:

Theorem 1.2.1 (Minkowski [1896], Steinitz [1916], Weyl [1935])
Let $P \subset \mathbb{R}^n$. Then P is a (rational) polytope if and only if P is a bounded (rational) polyhedron. $\square$

More generally:

Theorem 1.2.2 (Motzkin [1936])
Let $P \subset \mathbb{R}^n$. Then P is a (rational) polyhedron if and only if $P = Q + C$ where Q is a (rational) polytope in $\mathbb{R}^n$ and C is a (rational) finitely generated cone in $\mathbb{R}^n$. $\square$

Here a *finitely generated cone* is a set of the form $\{\sum_{i=1}^{n} \lambda_i x_i \mid \lambda_i \geqq 0\ (i=1,\ldots,m)\}$ with $x_1,\ldots,x_m \in \mathbb{R}^n$. (If $x_1,\ldots,x_n \in \mathbb{Q}^n$ we call the cone *rational*.). As usual $Q + C := \{q+c \mid q \in Q, c \in C\}$.

A *face* of a polyhedron P is a subset of the form $\{x \in P \mid a^T x = \beta\}$, where $a^T x \leqq \beta$ is valid for P. A face F of P is called *proper* if $F \neq P$. (Note that P and $\emptyset$ are faces of P.)

<u>Lemma 1.2.3</u>
Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. *Then* F *is a non-empty face of* $P := \{x \in \mathbb{R}^n \mid Ax \leqq b\}$ *if and only if* $F = \{x \in P \mid A_1 x = b_1\} \neq \emptyset$, *for some matrix* $[A_1 | b_1]$ *obtained from* $[A|b]$ *by deleting (zero or more) rows.* □

In other words: any face of a polyhedron P can be obtained by setting to equality some of the inequalities in the system defining P.

Of particular interest are the (inclusionwise) minimal nonempty faces of a polyhedron, and the maximal proper faces (the facets) of a polyhedron.

MINIMAL NONEMPTY FACES, VERTICES

Let $P := \{x \in \mathbb{R}^n \mid Ax \leqq b\}$. Let F be a minimal nonempty face of P. Then it can be shown that there exists a subsystem $A_1 x \leqq b_1$ of $Ax \leqq b$ such that $F = \{x \in \mathbb{R}^n \mid A_1 x = b_1\}$. (Clearly, we may assume the rows of A_1 to be linearly independent.) So a minimal nonempty face of P is an affine subspace of $\mathbb{R}^n$. If F contains a single vector, x_F say, then we call x_F a *vertex* of P. If <u>one</u> minimal nonempty face of P is a vertex then <u>each</u> minimal nonempty face of P is a vertex. In that case we call the polyhedron P *pointed.* (More generally, all minimal nonempty faces have the same affine dimension).

<u>Lemma 1.2.4</u>
Each nonempty polytope is pointed. Moreover, each polytope is the convex hull of its vertices. □

If $P = \{x \in \mathbb{R}^n | Ax \leqq b\}$ is a pointed polyhedron, then A is of full column-rank. Moreover $\hat{x}$ is a vertex of P if and only if $\hat{x} \in P$ and there exists a nonsingular n×n submatrix A_1 of A such that $\hat{x} = A_1^{-1} b_1$ (b_1 being the subvector of b corresponding to A_1).

FACETS

A *facet* of a polyhedron is an (inclusionwise) maximal nonempty proper face of P. There is a strong relation between the facets of a polyhedron and a defining system of linear inequalities of that polyhedron. To explain this relation we restrict ourselves to full-dimensional polyhedra. A polyhedron is *full-dimensional* if it is not contained in any hyperplane $\{x \in \mathbb{R}^n | a^T x = \beta\}$ $(a \in \mathbb{R}^n \backslash \{0\})$. Let P be a full-dimensional polyhedron, and $F_1, \ldots, F_s$ be its facets. Then there exists a system of inequalities $a_1^T x \leqq \beta_1, \ldots, a_s^T x \leqq \beta_s$ defining P such that $F_i = \{x \in P | a_i^T x = \beta_i\}$ for $i=1,\ldots,s$. Moreover any defining system $\hat{a}_1^T x \leqq \hat{\beta}_1, \ldots, \hat{a}_t^T x \leqq \hat{\beta}_t$ satisfies: for each $i=1,\ldots,s$ there exists a $j=1,\ldots,t$ and a $\lambda > 0$, such that $a_i = \lambda \hat{a}_j$ and $\beta_i = \lambda \hat{\beta}_j$. So the inequalities $a_1^T x \leqq \beta_1, \ldots, a_s^T x \leqq \beta_s$ essentially occur in any defining system of the polyhedron.

LINEAR PROGRAMMING

Linear programming means optimizing a linear functional over a polyhedron. A typical way to formulate a *linear programming problem* is:

$$(1.2.5) \quad \begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leqq b. \end{array}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. ("s.t." stands for "subject to".) Instead of maximizing $c^T x$ we also could consider minimizing $c^T x$.) Matrix A is called the *constraint matrix* of (1.2.5). An important result in linear programming is the so-called linear programming duality theorem:

Theorem 1.2.6 (von Neumann [1947]; Gale, Kuhn and Tucker [1951])
Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. *Then* $\max\{c^T x | Ax \leqq b\} = \min\{y^T b | y^T A = c^T,\ y \geqq 0\}$ *provided that both optimization problems have a feasible solution.* □

(Note that essentially Theorem 1.2.6 has the same content as Farkas Lemma (1.1.1).)

<u>Remarks</u>:

(i) If one of the two problems in Theorem 1.2.6 has a feasible solution then the optimum of that problem exists if and only if the other problem also has a feasible solution.

(ii) The minimization problem in Theorem 1.2.6 is called the *dual (linear programming) problem* of (1.2.5).
Any problem of type

$$(1.2.7)\quad \begin{aligned} \max\ & c_1^T x_1 + c_2^T x_2 + c_3^T x_3 \\ \text{s.t. } & A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \leqq b_1 \\ & A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2 \\ & A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \geqq b_3 \\ & x_1 \geqq 0, \qquad\qquad x_3 \leqq 0. \end{aligned}$$

can be seen as a special case of (1.2.5). The dual problem then is equivalent to:

$$(1.2.8)\quad \begin{aligned} \min\ & y_1^T b_1 + y_2^T b_2 + y_3^T b_3 \\ \text{s.t. } & y_1^T A_{11} + y_2^T A_{21} + y_3^T A_{31} \geqq c_1 \\ & y_1^T A_{12} + y_2^T A_{22} + y_3^T A_{32} = c_2 \\ & y_1^T A_{13} + y_2^T A_{23} + y_3^T A_{33} \leqq c_3 \\ & y_1 \geqq 0, \qquad\qquad y_3 \leqq 0. \end{aligned}$$

In case A, b and c are rational it easily follows from Theorem 1.2.6 that the duality theorem for linear programming forms a good characterization for the linear programming problem. So the linear programming problem is well-characterized. (The fact Theorem 1.2.6 gives a good characterization, follows from the fact that both (1.2.5) and its dual have optimal solutions x respectively y such that the size of x and y is bounded by a polynomial in the sizes of A, b and c. This follows from Cramer's rule (and Lemma 1.2.3).) Note that in essence the remarks above are the same as saying that LI is well-characterized, and has Farkas Lemma as a good-characterization (cf. Section 1.1).

The most prominent algorithm for linear programming is the simplex method due to Dantzig [1951]. This method turned out to be efficient in practice, but no version of it could be proved to be a polynomial-time algorithm. In fact, most versions are not (e.g., Klee and Minty [1972]). A polynomial-time algorithm for linear programming is the so-called ellipsoid method Khachiyan [1979], cf. Grötschel, Lovász, and Schrijver [1987]). This algorithm is based on the ellipsoid method for nonlinear programming by Shor [1970a,b,1977] and Yudin, and Nemirovskii [1976]. Besides settling the longstanding open problem whether or not linear programming is polynomially solvable, the ellipsoid method has important implications for combinatorial optimization. Later we will come back to these implications (due to Grötschel, Lovász and Schrijver [1981]).

To explain the importance of polyhedral theory for combinatorial optimization we next consider as a typical example the matching problem.

EXAMPLE: THE MATCHING PROBLEM

Let G be a graph. (For graph terminology, see Section 1.3.) A *matching* in G is a subset M of E(G) such that each $u \in V(G)$ is endpoint of at most one edge in M. The weighted matching problem is:

(1.2.9) Given $c \in \mathbb{Z}^{E(G)}$, find a matching in G such that $\sum_{e \in M} c_e$ is maximal.

This problem can be reformulated as an integer linear programming problem:

$$(1.2.10)\quad \max\ c^T x \quad \text{s.t.} \quad \begin{array}{ll} x_e \geqq 0 & (e \in E(G)); \\ \sum_{e \in \delta(u)} x_e \leqq 1 & (u \in V(G)); \\ x_e \in \mathbb{Z} & (e \in E(G)). \end{array}$$

We would like to use the polynomial-time solvability of linear programming to solve (1.2.9). The first approach is to solve the *linear programming relaxation* of (1.2.10) (obtained by dropping the integrality conditions). However typically we will find a non-integral optimal solution. The reason is that not all the vertices of

$$Q := \{x \in \mathbb{R}^{E(G)} \mid x_e \geqq 0,\ e \in E(G);\ \sum_{e \in \delta(u)} x_e \leqq 1,\ u \in V(G)\}$$

are integral. All vertices of Q are integral if and only if G is bipartite (Birkhoff [1946], von Neumann [1953]). So if G is non-bipartite the linear programming relaxation may not solve the original problem.

This problem does not arise if we define

$$P_{\mathcal{M}} := \operatorname{conv}\{x_M \in \mathbb{R}^{E(G)} \mid M \text{ is a matching}\},$$

and formulate (1.2.9) as $\max\{c^T x \mid x \in P_{\mathcal{M}}\}$. The latter is a linear programming problem, as $P_{\mathcal{M}}$ is a polytope (the *matching polytope*), and hence a polyhedron (Theorem 1.2.1). However to apply linear programming techniques, we need a description of $P_{\mathcal{M}}$ in term of inequalities. Edmonds [1965c] showed that the following is such a description:

$$(1.2.11)\quad \begin{array}{ll} x_e \geqq 0 & e \in E(G); \\ \sum_{e \in \delta(u)} x_e \leqq 1 & u \in V(G); \\ \sum_{e \subset U} x_e \leqq \frac{|U|-1}{2} & U \subseteq V(G),\ |U| \geqq 3 \text{ and odd}. \end{array}$$

So in principle we can solve (1.2.9) as a linear programming problem. But if we give (1.2.11) as an input to any linear programming algorithm we encounter a new difficulty. This input is far to large. The number of inequalities of the third type in (1.2.11) is exponential in the size of the original problem (1.2.9). (And if G is a complete graph all these inequalities correspond to facets.) However, this difficulty is not so serious. Edmonds [1965c] avoided it by writing down during any stage of his algorithm only $|E(G)|$ of the inequalities in (1.2.11) explicitly (Note that any vertex of the polyhedron of dual feasible solutions has at most $|E(G)|$ non-zero variables.) Edmonds' algorithm for the weighted matching problem is a polynomial-time algorithm. There is another way to avoid writing down all the inequalities in (1.2.11) explicitly. Padberg and Rao [1982] gave a polynomial-time separation algorithm for $P_{\mathcal{M}}$. A *separation algorithm* for a polyhedron $P \subset \mathbb{R}^n$ is an algorithm for the following *separation problem* for P:

(1.2.12) Given $\hat{x} \in \mathbb{R}^n$, decide whether or not $\hat{x} \in P$. If not find an inequality $a^T x \leqq \beta$, valid for P, such that $a^T \hat{x} > \beta$.

A nice feature of the ellipsoid method is, that instead of a complete list of inequalities for a polyhedron, it needs only a separation algorithm for the polyhedron, in order to optimize over it. If the separation algorithm is polynomial-time, the optimization algorithm thus obtained is a polynomial-time algorithm too (Grötschel, Lovász, and Schrijver [1981]). It is particularly important for combinatorial problems, as the related polyhedra, like the matching polytope, typically have many facets. (For combinatorial applications see also Grötschel, Lovász, and Schrijver [1981].) Moreover Grötschel, Lovász, and Schrijver showed that in fact the existence of an polynomial-time algorithm for optimizing over a class of polyhedra is equivalent to the existence of an polynomial-time separation algorithm for the class of polyhedra.

INTEGRAL POLYHEDRA

An *integer linear programming problem* is an optimization problem of the following form:

$$(1.2.13) \quad \max \; c^T x \quad \text{s.t. } Ax \leqq b, \; x \in \mathbb{Z}^n.$$

Many combinatorial optimization problems can be formulated as integer linear programming problems. However, integer linear programming problems are generally hard to solve. In fact, integer linear programming is $\mathcal{NP}$-hard (Cook [1971]). It is polynomially solvable if the number of variables is fixed (Lenstra [1983]). As in the example above, one can try to apply linear programming techniques. Let P be a polyhedron. Then the *integer hull* of P is the convex set $P_I := \text{conv}(P \cap \mathbb{Z}^n)$.

<u>Theorem 1.2.14</u> (Meyer [1974])
Let P *be a rational polyhedron in* $\mathbb{R}^n$. *Then* P_I *is a rational polyhedron.* □

The problem of finding a system of inequalities defining P_I will be discussed in Section 2.1. In this section we restrict ourselves to the case $P_I = P$. We call a polyhedron $P \subset \mathbb{R}^n$ *integral* if $P = P_I$. Equivalently, a polyhedron is integral if and only if P is rational and each minimal face contains an integral vector. In particular, if P is pointed, then P is integral if and only if P is rational and all its vertices are integral vectors.

Of particular interest for integral polyhedra are totally unimodular matrices. A matrix is called *totally unimodular* if all its subdeterminants are 0,1 or -1. So, in particular, all the entries of a totally unimodular matrix are 0,1 or -1. The following result is well-known:

<u>Theorem 1.2.15</u> (Hoffman and Kruskal [1956])
Let $A \in \mathbb{Z}^{m \times n}$. *Then the following are equivalent:*
(i) $\{x \in \mathbb{R}^n \mid a \leqq x \leqq b, \; c \leqq Ax \leqq d\}$ *is integral for each* $a,b \in \mathbb{Z}^n$; $c,d \in \mathbb{Z}^m$;
(ii) A *is totally unimodular.* □

We also want to mention a version of this theorem which is perhaps not so well-known: We call a matrix $A \in \mathbb{Z}^{m \times n}$ *unimodular* if for each matrix B consisting of r linearly independent columns of A (r:= r(A), the rank of A), the greatest common divisor of all r × r subdeterminants of B is equal to 1.

Theorem 1.2.16 (Hoffman and Kruskal [1956])
Let $A \in \mathbb{Z}^{m \times n}$. *Then the following are equivalent:*
(i) *For each* $b \in \mathbb{Z}^m$, $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ *is integral;*
(ii) *For each* $c \in \mathbb{Z}^n$, $\{y \in \mathbb{R}^n \mid y^T A = c^T,\ y \geq 0\}$ *is integral;*
(iii) A^T *is unimodular.* □

Both theorems characterize classes of constraint matrices for which certain polyhedra are integral. The following theorem gives a characterization for a fixed polyhedron to be integral. (We come back to totally unimodular matrices in Section 1.4.)

Theorem 1.2.17 (Edmonds and Giles [1977])
A rational polyhedron P *is integral, if and only if each rational supporting hyperplane of* P *contains an integral vector.* □

Here a *rational supporting hyperplane* in $\mathbb{R}^n$, is a subset H = $\{x \in \mathbb{R}^n \mid a^T x = \beta\}$ with $a \in \mathbb{Z}^n \setminus \{0\}$, $\beta \in \mathbb{Q}$, such that $H \cap P \neq \emptyset$ and $ax \leq \beta$ is valid for P.

Theorem 1.2.15 can be reformulated as the following corollary.

Corollary 1.2.18
Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$. *Then the following are equivalent:*
(i) $\{x \in \mathbb{Q}^n \mid Ax \leq b\}$ *is integral.*
(ii) For each $c \in \mathbb{Z}^n$, *for which* $\max\{c^T x \mid Ax \leq b\}$ *exists, we have* $\max\{c^T x \mid Ax \leq b\} \in \mathbb{Z}$. □

Later in this monograph, we use the following version of Corollary 1.2.18.

Corollary 1.2.19
Let $A \in \mathbb{Q}^{m \times n}$, $B \in \mathbb{Q}^{m \times k}$; $b \in \mathbb{Q}^m$. *Then the following are equivalent.*

(i) *For each* $c \in \mathbb{Z}^m$, *for which* $\max\{c^T x \mid Ax + By \leqq b\}$ *exists, we have* $\max\{c^T x \mid Ax + By \leqq b\} \in \mathbb{Z}$.

(ii) *For each* $c \in \mathbb{Z}^m$, *for which* $\max\{c^T x \mid Ax + By \leqq b\}$ *exists, there exists an optimal solution* $(\hat{x}, \hat{y}) \in \mathbb{R}^m \times \mathbb{R}^k$ *with* $\hat{x} \in \mathbb{Z}^m$.

Proof (that Corollary 1.2.19 follows from Corollary 1.2.18): Define $P := \{x \in \mathbb{R}^n \mid \exists_{y \in \mathbb{R}^k} [Ax + By \leqq b]\}$. Then P is a rational polyhedron. The equivalence to be proved is exactly the equivalence in Corollary 1.2.18 for P. □

A system of inequalities $Ax \leqq b$, with $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, is called *totally dual integral* if the minimum in

$$\max\{c^T x \mid Ax \leqq b\} = \min\{y^T b \mid y \geqq 0, \ y^T A = c^T\}$$

has an integral optimal solution for each $c \in \mathbb{Z}^n$ for which the minimum exists. The following theorem directly follows from Corollary 1.2.18.

Theorem 1.2.20 (Edmonds and Giles [1977])
Let $Ax \leqq b$, *be a totally dual integral system of inequalities. If* b *is integral, then* $\{x \in \mathbb{R}^n \mid Ax \leqq b\}$ *is integral.* □

Not any system defining an integral polyhedron is totally dual integral. Indeed, $\{(x_1, x_2)^T \in \mathbb{R}^2 \mid 2x_1 + x_2 \leqq 2, \ x_1 \geqq 0, \ x_2 \geqq 0\}$ is an integral polyhedron. However $\max\{3x_1 + x_2 \mid 2x_1 + x_2 \leqq 2, \ x_1 \geqq 0, \ x_2 \geqq 0\} = 3$. Hence the dual problem $\min\{2y_1 \mid 2y_1 - y_2 = 3, \ y_1 - y_3 = 1; \ y_1, \ y_2, \ y_3 \geqq 0\}$ has no integral optimal solution. On the other hand:

Theorem 1.2.21 (Giles and Pulleyblank [1979], Schrijver [1981])
Let $P \subset \mathbb{R}^n$ *be a rational polyhedron.*

(i) *(Giles and Pulleyblank) There exists a totally dual integral system* $Ax \leqq b$, *with* $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Q}^m$ *such that* $P = \{x \in \mathbb{R}^n \mid Ax \leqq b\}$. *Moreover,* b *can be chosen integral if and only if* P *is integral.*

(ii) (Schrijver) If P *is full dimensional, then there exists a unique minimal totally dual integral system* $Ax \leqq b$ *with* $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Q}^m$; *and* $P = \{x \in \mathbb{R}^n \mid Ax \leqq b\}$. *Moreover* $b \in \mathbb{Z}^m$ *if and only if* P *is integral.*

$\square$

1.3. GRAPHS AND SIGNED GRAPHS

We assume the reader to be familiar with the basic notions and results of graph theory (cf. Bondy and Murty [1976], Wilson [1972]). Below we give some notational conventions and basic definitions. We denote the node-set of an (undirected) graph G by V(G), and the edge-set by E(G). We allow loops and parallel edges. A graph with no loops and parallel edges, G is called *simple*. An edge e connecting u and v is typically denoted by uv, u and v are called the *endpoints* of uv. We call u and uv *incident*. And we call u and v *adjacent* if $uv \in E(G)$.

We assume the following notions to be known: *path*; *(spanning) tree* and *forest*; *bipartite*; *complete* (the complete simple graph on n nodes is denoted by K_n); *complete bipartite* (the complete bipartite simple graph with *colour classes* of size n and m is denoted by $K_{n,m}$); *connected; component; graph isomorphism* (denoted by "$\sim$"); *graph homeomorphism; subgraph; deletion* and *contraction* (the graph obtained from G by deleting (contracting) edge e is denoted by $G\backslash e$ (G/e respectively)); *induced* subgraph ($G|U$ denotes the subgraph of G induced by $U \subset V(G)$); *planar graph,* a *planar dual* of a planar graph G (a planar dual is denoted by G^*).

We want to distinguish between the notions circuit and cycle. A *circuit* of length k is a graph C with $V(C) = \{v_0, v_1, \ldots, v_{k-1}\}$ ($v_i \neq v_j$ if $i \neq j$) and $E(C) = \{v_0v_1, v_1v_2, \ldots, v_{k-2}v_{k-1}, v_{k-1}v_0\}$. A *cycle* is a graph in which all degrees are even (the *degree* of a node u is the number of edges with endpoint u). If $U \subset V(G)$, then $\delta(U) := \{uv \in E(G) \mid u \in U,\ v \in V(G)\backslash U\}$ is the *coboundary* of U.

The node-set of a directed graph D, is denoted by V(D), its arc-set by A(D) an arc going from u (the *tail* of the arc) to v (the *head* of the arc) is typically denoted by $\overrightarrow{uv}$ or $\overleftarrow{vu}$. We allow loop-arcs ($\overrightarrow{uu}$) as well as parallel arcs. ($\overrightarrow{uv}$ and $\overrightarrow{vu}$ are <u>not</u> considered to be parallel). Terms like *directed path*, and *directed circuit* are assumed to be familiar to the reader.

The *node-edge incidence matrix* $M_G \in \mathbb{R}^{V(G)\times E(G)}$ of a graph G is defined by

$$(M_G)_{u,e} = \begin{cases} 1 & e = uv, \text{ for some } v \neq u \\ 2 & e = uu \\ 0 & \text{else} \end{cases}$$

for each $u \in V(G)$, $e \in E(G)$. The *edge-node incidence matrix* of G is M_G^T. The *node-arc incidence matrix* $N_D \in \mathbb{R}^{V(D) \times A(D)}$ of a directed graph D is defined by

$$(N_D)_{u,a} = \begin{cases} 1 & a = \overrightarrow{vu}, \text{ for some } v \neq u \\ -1 & a = \overrightarrow{uv}, \text{ for some } v \neq u \\ 0 & \text{else.} \end{cases}$$

for each $u \in V(D)$, $a \in A(D)$. The *arc-node incidence matrix* of D is N_D^T. A function $f \in \mathbb{R}^{A(D)}$ with $N_D f = 0$ is called a *circulation* in D.

Let G be an undirected graph. A (k-)*node cutset* of G is a set $U \subset V(G)$, (with $|U| = k$ and) such that $G|(V(G) \setminus U)$ is not connected. In that case G has two subgraphs G_1, G_2 with the following properties:

$V(G_1) \cap V(G_2) = U$; $V(G_1) \cup V(G_2) = V(G)$; $V(G_1) \neq U \neq V(G_2)$;

$E(G_1) \cap E(G_2) = \emptyset$; $E(G_1) \cup E(G_2) = E(G)$.

We call two such graphs G_1 and G_2 *the two sides of the cutset* U. (Note that G_1 and G_2 need not be uniquely determined. If several choices are possible we just choose G_1 and G_2 arbitrarily.)

G is k-*connected* if G has no ℓ-node cutset with $\ell < k$. If U is a node cutset and $S, T \subset V(G)$, we say that U *separates* S and T if $U \cap S = \emptyset = U \cap T$ and no component of $G|(V(G) \setminus U)$ contains elements both from S and from T. The following result is used several times throughout this monograph.

Theorem 1.3.1 (Menger [1927])
Let G *be a graph, and* $s, t \in V(G)$, *such that* $st \notin E(G)$. *Then the maximum number pairwise internally node disjoint paths from* s *to* t *is equal to the minimum cardinality of a node-cut set separating* s *and* t. $\square$

Here, two paths P_1 and P_2 from s to t are *internally node disjoint* if $V(P_1) \cap V(P_2) = \{s,t\}$. There are many versions of Menger's Theorem (cf. Schrijver [1983], Reichmeider [1984]). One of these versions is the well-known max-flow min-cut theorem of Ford and Fulkerson [1956].

Another result we use several times in this monograph is:

Theorem 1.3.2

Let D *be a directed graph, and* $\ell \in \mathbb{R}^{A(D)}$. *Then the following are equivalent*

(i) $\sum_{a \in A(C)} \ell_a \geqq 0$ *for each directed circuit* C *in* D;

(ii) There exists a $\pi \in \mathbb{R}^{V(D)}$ *with* $\pi_v - \pi_u \leqq \ell_{\overrightarrow{uv}}$ *for each* $\overrightarrow{uv} \in A(D)$.

If ℓ *in (i) is integer valued, then* π *in (ii) can also be taken integer valued.* □

SIGNED GRAPHS

A *signed graph* is a pair (G,Σ), where Σ is a subset of the edge set $E(G)$ of G. The edges in Σ are called *odd*, the other edges *even*. A circuit C in G is called *odd* (*even*, respectively) if $E_0 \cap E(C)$ is odd (even, respectively) we call a finite set X *odd* if $|X|$ is odd.) We call a signed graph *bipartite* if $\Sigma = \delta(U)$ for some $U \subset V(G)$. For example $(G,\emptyset)$ is bipartite. Moreover, $(G,E(G))$ is bipartite if and only if G is a bipartite graph in the usual sense. It is easy to see that a signed graph is bipartite if and only if it contains no odd circuits. Let (G,Σ) be a signed graph, and let $U \subset V(G)$. Obviously (G,Σ) and $(G,\Sigma\Delta\delta(U))$ have the same collection of odd circuits (Δ denotes the set-theoretic *symmetric difference*). We call the operation $\Sigma \to \Sigma\Delta\delta(U)$ *resigning* (*on* U). We call two signed graphs (G,Σ) and (G',Σ') *equivalent* (notation: $(G,\Sigma) \sim (G',\Sigma')$) if there exists a set $U \subset V(G)$, and a bijection φ from $V(G)$ to $V(G')$ and a bijection ψ from $E(G)$ to $E(G')$ such that

(i) e is an edge from u to v in G, if and only if $\psi(e)$ is an edge from $\varphi(u)$ to $\varphi(v)$ in G'.

(ii) $\psi[\Sigma\Delta\delta(U)] = \Sigma'$.

We say that (G,Σ) *reduces to* (G',Σ') if (G',Σ') can be obtained from (G,Σ) by a series of the following operations:

- deleting an edge from G (and from Σ).
- contracting an even edge in G.
- resigning.

In this monograph a central role is played by the signed graph indicated in Figure 1.1. Wriggled lines stand for pairwise openly disjoint paths, each containing at least one edge; the term **odd** in a face indicates that the bounding circuit is an odd circuit. We call such a signed graph an *odd-K_4*.

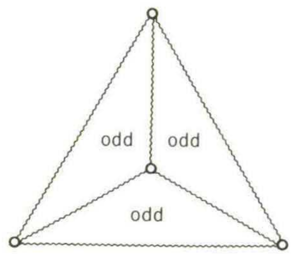


Figure 1.1

An example of an odd-K_4 is $\widetilde{K}_4 := (K_4, E(K_4))$ where K_4 is the complete graph on four nodes.

Remark:

(G,Σ) is an odd-K_4 if and only if it can be constructed by the following operations:

- resign $\widetilde{K}_4$ to a signed graph (K_4,Σ');
- then replace each edge e in K_4 by a path P_e (this yields G);
- finally choose $\Sigma \subseteq E(G)$ such that for each $e \in E(K_4)$: $|\Sigma \cap E(P_e)|$ is odd if and only if $e \in \Sigma'$.

The following is easy to prove.

Lemma 1.3.3

Let (G,Σ) be a signed graph. Then (G,Σ) contains an odd-K_4 as a subgraph if and only if (G,Σ) reduces to $\widetilde{K}_4$. $\square$

We next show a technical Lemma, which will be used in Chapter 2 (Theorem 2.3.3) and in Chapter 4 (Theorem 4.2.2) (see also Gerards [1987]).

<u>Lemma 1.3.4</u>

Let (G,Σ) be a signed graph with no odd-K_4 as a subgraph, and with no one-node cutset. Let C be a non-separating odd circuit in G with $C\neq G$. If C satisfies:

(i) $V(C)\cap V(C') \neq \emptyset$ *for each odd-circuit* C' *in* (G,Σ);

(ii) C *contains at least three nodes with degree at least three,*

then C *has a subgraph* I_C *such that:*

(i') I_C *is a path,* $V(I_C) \neq \emptyset$;

(ii') Any odd circuit C' *in* (G,Σ) *contains* I_C *as a subgraph;*

(iii') There exists an odd circuit C' *in* (G,Σ) *such that* $V(C)\cap V(C') = V(I_C)$ *and* $E(C)\cap E(C') = E(I_C)$. □

Before we prove Lemma 1.3.4 we explain the notion: "non-separating circuit".

Let G be a graph, and C a circuit. We call two edges e, $f\in E(G)\setminus E(C)$ *equivalent with respect to* C if e=f or there exists a path v_0v_1, $v_1v_2,\ldots v_{k-1}v_k$, with $v_0v_1 = e$, $v_{k-1}v_k = f$ and $v_1,\ldots,v_{k-1}\notin V(C)$. The equivalence classes of this equivalence relation are called the *bridges* of C (In particular, a chord uv of C (i.e. u, $v\in V(C)$, $uv\notin E(C)$), forms a bridge of C.). A circuit C is called *non-separating* in G if it has at most one bridge. If C has more than one bridge, C is called *separating*.

<u>Proof of Lemma 1.3.4</u>

Clearly $V(G)\setminus V(C) \neq \emptyset$. (If $V(G) = V(C)$ then C has exactly one chord, uv say, as $C\neq G$ and C is non-separating. Now for I_C we can take one of the two paths on C from u to v.) Let T be a tree spanning $V(G)\setminus V(C)$ (which exists, as C is non-separating). Now delete all the edges contained in $V(G)\setminus V(C)$ which are not in T. Resign such that $\Sigma\cap E(T) = \emptyset$, and then contract the edges in T. As the edges contained in $V(G)\setminus V(C)$ form a bipartite graph (by condition (i)), each odd circuit in the original signed graph contains an odd circuit in the reduced signed graph. Conversely each odd circuit in the contracted signed graph is contained in an odd circuit of the original signed graph. Hence we may assume that (G,Σ) is the contracted graph, i.e., $V(G) = V(C)\cup\{w\}$ for some node w.

Let C' be an odd circuit in G which has a minimum number of edges in common with C. Define I_C by $V(I_C) = V(C)\cap V(C')$ and $E(I_C) = E(C)\cap E(C')$. Obviously I_C satisfies (i') and (iii'). Suppose (ii') is not satisfied by I_C. Let C'' be an odd circuit not containing I_C. By the minimality of $|E(C')\cap E(C)|$, we have that $E(C')\cap E(C)\cap E(C'') = \emptyset$. Now there are five possibilities indicated in Figure 1.2 below:

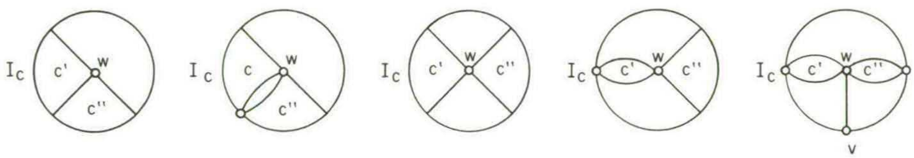


Figure 1.2

In each of them, (G,Σ) contains an odd-K_4. The existence of edge wv in the right most figure above follows from (ii). □

1.4. BINARY MATROIDS = BINARY SPACES

All matroid theory we need in this monograph concerns binary matroids. Therefore all notions will be defined for binary matroids only. For the theory of matroids in general we refer to Welsh [1976]. Mostly we use the terminology of Welsh's book.

Let E be a finite set. We consider the set $GF(2)^E$ in the obvious way as a linear space over GF(2). A *binary space on* E is a linear subspace of $GF(2)^E$. So, in particular, $GF(2)^E$ is a binary space on E.

A *binary matroid* $\mathcal{M}$ consists of a finite set $E = E(\mathcal{M})$ and a *binary space*, $\mathcal{C}(\mathcal{M})$, on E. We call $\mathcal{C}(\mathcal{M})$ the *cycle space* of $\mathcal{M}$. An alternative definition is: a binary matroid is a collection of subsets of a finite set closed under symmetric differences. Obviously these two definitions are equivalent. It will be convenient to intertwine the algebraic terminology of the first definition with the set-theoretic terminology of the second definition. We shall do this without explicitly specifying which terminology we use.

CYCLES, CIRCUITS AND INDEPENDENT SETS, THE DUAL MATROID

A member of $\mathcal{C}(\mathcal{M})$ is called a *cycle* of $\mathcal{M}$. Inclusionwise minimal non-empty cycles are called *circuits*. Each cycle can be partitioned into circuits.

A set $E' \subset E$ is called *independent* if E' contains no circuit. The *dual matroid* $\mathcal{M}^*$ of $\mathcal{M}$ is defined by $E(\mathcal{M}^*) := E(\mathcal{M})$ and $\mathcal{C}(\mathcal{M}^*) := \mathcal{C}(\mathcal{M})^\perp$ (where $V^\perp := \{x \mid x^T y = 0 \text{ for each } y \in V\}$).

If $\{e\}$ is a circuit, we call e a *loop*. If $\{e,f\}$ is a circuit, e and f are called *parallel*. A *co-cycle* in $\mathcal{M}$ is a cycle in $\mathcal{M}^*$. Similarly we use the terms *co-circuit* and *co-loop*. If e and f are parallel in $\mathcal{M}^*$, we say that e and f are *in series* in $\mathcal{M}$.

BINARY REPRESENTATION

A matrix M with rows from $GF(2)^E$ is called a (*binary*) *representation* of the binary matroid $\mathcal{M}$ if $\mathcal{C}(\mathcal{M}) = \mathcal{N}(M) := \{x|Mx = 0\}$. We also say: M *represents* $\mathcal{M}$ *over* GF(2).

Denote the submatrix of M consisting of the columns indexed by $E' \subset E$ by $M|E'$. Then the *rank* $r_{\mathcal{M}}(E')$ of E' is the rank of $M|E'$. Clearly, set E' is independent if and only if $r_{\mathcal{M}}(E') = |E'|$. Obviously, 'rank' does not depend on the actual representation M, as $r_{\mathcal{M}}(E')$ is equal to the maximum cardinality of an independent subset of E'. (Note that, by Steinitz' exchange theorem for linear spaces, all inclusionwise maximal independent subsets of E' have the same cardinality.)

BASIC, STANDARD REPRESENTATION

A *basis* of $\mathcal{M}$ is an inclusion wise maximal independent set of $E(\mathcal{M})$. All bases have the same cardinality, namely $r_{\mathcal{M}}(E(\mathcal{M}))$, called the *rank* of $\mathcal{M}$. Let $\mathcal{M}$ be a binary matroid of rank r, and let $\mathcal{B}$ be a basis of $\mathcal{M}$. Then the *standard representation* of $\mathcal{M}$ *over* GF(2) with respect to $\mathcal{B}$ is the (unique) representation $[I_r|A]$ of $\mathcal{M}$, where I_r is the $r\times r$-identity matrix, and the columns of I_r correspond to the elements in $\mathcal{B}$. (From now on we delete the subscript r from I_r.)

<u>Lemma 1.4.1</u>
Let $\mathcal{M}$ *be a binary matroid with standard representation* $[I|A]$. *Then* $[A^T|I]$ *is a standard representation of* $\mathcal{M}^*$.

<u>Proof</u>: $\mathcal{N}([I|A])^{\perp} = \mathcal{N}([A^T|I])$. □

From this we immediately see that the bases of $\mathcal{M}^*$ are exactly the complements of the bases of $\mathcal{M}$.

BASIS-EXCHANGE, PIVOTING

How can we go from one standard representation to any other? The answer is: by a series of pivots. Let A be a matrix over a field **F**. In this monograph *pivoting* A *on* an entry $\varepsilon \neq 0$ of A over **F** means replacing

$$(1.4.2)\quad A = \left[\begin{array}{c|c} \varepsilon & y^{\mathsf{T}} \\ \hline x & D \end{array}\right] \text{ by } B = \left[\begin{array}{c|c} -\varepsilon & y^{\mathsf{T}} \\ \hline x & D - \varepsilon^{-1} x y^{\mathsf{T}} \end{array}\right]$$

(ε is called the *pivot element*. The specific position of ε in (1.4.2) is just an example. The pivot element can be anywhere in A. The row (column) of A containing E is called the *pivot row* (*column*).) Now let $\mathcal{B}$ be a basis of a binary matroid $\mathcal{M}$ with standard-representation $M = [I|A]$. Index the rows of M, by the elements of $\mathcal{B}$ such that the ones in I are exactly in the positions M_{ee} ($e \in \mathcal{B}$). Let $e \in \mathcal{B}$, and $f \notin \mathcal{B}$. Then $(\mathcal{B} \setminus \{e\}) \cup \{f\}$ is a basis if and only of $A_{ef} = 1$. If $A_{ef} = 1$ pivoting A on A_{ef} and interchanging the column indices e and f, yields a standard representation of $\mathcal{M}$ with respect to the basis $(\mathcal{B} \setminus \{e\}) \cup \{f\}$.

Example:

$$\begin{array}{cc} & \\ e & \\ & \\ & \end{array}\!\!\begin{array}{c} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & \underline{1} \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \\ \begin{array}{cccccc} & e & & & & f \end{array} \end{array} \xrightarrow{\text{pivot}} \begin{array}{c} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \\ \begin{array}{cccccc} & f & & & & e \end{array} \end{array}.$$

The fact that any standard representation can be transformed to any other standard representation by a series of pivots, follows from the following well-known "basis exchange" property: If $\mathcal{B}$ and $\mathcal{B}'$ are two basis of a (binary) matroid then for each $f \in \mathcal{B}' \setminus \mathcal{B}$ there exists an $e \in \mathcal{B} \setminus \mathcal{B}'$ such that $(\mathcal{B} \setminus \{e\}) \cup \{f\}$ is a basis too.

MINORS

Let $\mathcal{M}$ be a binary matroid, and let $e \in E := E(\mathcal{M})$. The matroid $\mathcal{M}\backslash e$ obtained from $\mathcal{M}$ by *deleting* e is defined by $E(\mathcal{M}\backslash e) := E\backslash\{e\}$, and $\mathcal{C}(\mathcal{M}\backslash e) := \{C \subset E\backslash\{e\} \mid C \in \mathcal{C}(\mathcal{M})\}$. The matroid $\mathcal{M}/e$ obtained from $\mathcal{M}$ by *contracting* e is defined by $E(\mathcal{M}/e) = E\backslash\{e\}$ and $\mathcal{C}(\mathcal{M}/e) := \{C \subset E\backslash\{e\} \mid C \in \mathcal{C}(\mathcal{M})$ or $C \cup \{e\} \in \mathcal{C}(\mathcal{M})\}$.

Algebraically, deleting e from $\mathcal{M}$ means taking the binary space obtained by intersecting $\mathcal{C}(\mathcal{M})$ with the hyperplane $x_e = 0$ (and then deleting the component x_e form all vectors x). Contracting e from $\mathcal{M}$ can be interpreted algebraically by projecting $\mathcal{C}(\mathcal{M})$ on the hyperplane $x_e = 0$ (and again removing component x_e from all vectors x). The following are easy to prove: $\mathcal{M}\backslash e = (\mathcal{M}^*/e)^*$, $\mathcal{M}/e = (\mathcal{M}^*\backslash e)^*$, and $\mathcal{M}\backslash e = \mathcal{M}/e$ if and only if e is a loop or a co-loop in $\mathcal{M}$.

We call a matroid resulting from $\mathcal{M}$ by a series of deletions and contractions a *minor* of $\mathcal{M}$. (Note that the order in which the deletions and contractions are carried out does not effect the resulting minor.) How to carry out deletion and contraction on a representation, M say, of a binary matroid $\mathcal{M}$? Deleting an element e from $\mathcal{M}$ corresponds to just deleting the column, m_e say, indexed by e from $\mathcal{M}$. Contracting e amounts to taking a non-zero entry M_{ie} in m_e, pivoting M on M_{ie}, and deleting the pivot row (indexed by i) and the pivot column m_e from the resulting matrix.

Two binary matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ are called *isomorphic* (notation: $\mathcal{M}_1 \sim \mathcal{M}_2$) if there exists a bijection φ: $E(\mathcal{M}_1) \longrightarrow E(\mathcal{M}_2)$ such that $\mathcal{C}(\mathcal{M}_2) = \{\varphi[C] \mid C \in \mathcal{C}(\mathcal{M}_1)\}$.

Let $\mathcal{M}$ and $\mathcal{M}'$ be binary matroids, and $x \in E(\mathcal{M})$. Then by saying "*$\mathcal{M}$ has no $\mathcal{M}'$-minor using* x" we mean: there are no sets E_1, $E_2 \subset E(\mathcal{M})$ such that $x \notin E_1 \cup E_2$ and $\mathcal{M}' \sim \mathcal{M}\backslash E_1/E_2$.

GRAPHIC MATROIDS

The terminology used above is somewhat hybrid. Terms like "independence" obviously come from linear algebra, whereas terms like 'circuit' and 'cycle' remind of graphs. We show that graphs indeed yield binary matroids.

Let M_G be the node-edge incidence matrix of an undirected graph G. Then the binary matroid represented over GF(2) by M_G (considered as a binary matrix), denoted by $\mathcal{M}(G)$, is called the *circuit matroid* of G. The circuits and cycles in G are exactly the circuits and cycles in $\mathcal{M}(G)$. The dual, $\mathcal{M}^*(G)$ of $\mathcal{M}(G)$ is called the *co-circuit* or *coboundary matroid* of G. The cycles of $\mathcal{M}^*(G)$ are exactly the coboundaries in G.

Let $\mathcal{M}$ be a binary matroid. If $\mathcal{M}$ is isomorphic to $\mathcal{M}(G)$ for some undirected graph G, then we call $\mathcal{M}$ *graphic*. If $\mathcal{M} \sim \mathcal{M}^*(G)$ for some G then $\mathcal{M}$ is called *co-graphic*. Obviously, if $G_1 \sim G_2$ then $\mathcal{M}(G_1) \sim \mathcal{M}(G_2)$. The converse is generally not true. However, Whitney [1932] proved that if G_1 is 3-connected, then $G_1 \sim G_2$ if and only if $\mathcal{M}(G_1) \sim \mathcal{M}(G_2)$. $\mathcal{M}(G)$ is co-graphic if and only if G is planar (Whitney [1933]).

REGULAR MATROIDS

A binary matroid $\mathcal{M}$ is called *regular* if and only if there exists a matrix N with rows from $\mathbb{R}^{E(\mathcal{M})}$ such that independence of elements in $\mathcal{M}$ is equivalent to independence over $\mathbb{R}$ of the corresponding columns in N. We call such an N a *real representation* of $\mathcal{M}$, or a *representation of $\mathcal{M}$ over* $\mathbb{R}$. For any basis $\mathcal{B}$ in a regular matroid $\mathcal{M}$ there exists a *real standard representation* $[I|A]$ of $\mathcal{M}$ with respect to $\mathcal{B}$ (so the columns of I correspond to the elements of $\mathcal{B}$).

Theorem 1.4.3
The dual as well as each minor of a regular matroid is regular.

Proof: If $[I|A]$ is a real standard representation of a regular matroid $\mathcal{M}$, then $[A^T|I]$ is a real standard representation of $\mathcal{M}^*$. Moreover, deleting an element of a regular matroid obviously yields a regular matroid. So each minor of a regular matroid is regular. (Note that $\mathcal{M}/e = (\mathcal{M}^*\backslash e)^*$.)

$\square$

GRAPHIC MATROIDS ARE REGULAR

Let G be an undirected graph, with node-edge incidence matrix $\mathcal{M}(G)$. Orient the edges of G in an arbitrary way (i.e. replace each edge by a directed arc). Denote the directed graph thus obtained by D. Now it is easy to see that the node-arc incidence matrix N_D of D is a real representation of $\mathcal{M}(G)$. Hence graphic matroids, and co-graphic matroids, are regular.

NON-REGULAR MATROIDS, THE FANO-PLANE

Not all binary matroids are regular. Indeed, consider the well-known Fano-plane

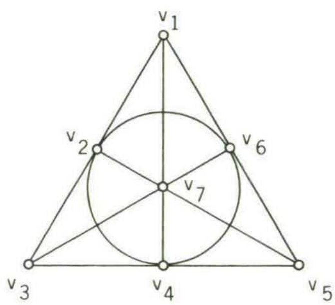


Figure 1.3

We call a collection V of points from $\{v_1,\ldots,v_7\}$ independent if $|V| \leq 2$ or $|V| = 3$ and the three points in V are not on one line of the Fano-plane (cf. Figure 1.3). This independence defines a binary matroid, denoted by F_7. A standard representation of F_7 is

$$[I|M(F_7)] := \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right].$$

A representation of F_7 over $\mathbb{R}$ would imply that the configuration in Figure 1.3 could be drawn in the euclidean plane with straight lines only. As we all know this is impossible. So F_7 is not regular.

TUTTE'S CHARACTERIZATION

Tutte proved that in a sense F_7 is the only non-regular binary matroid.

Theorem 1.4.4 (Tutte [1958])
Let $\mathcal{M}$ be a binary matroid. Then $\mathcal{M}$ is regular if and only if $\mathcal{M}$ has neither F_7 nor F_7^* as a minor. □

To keep the exposition transparant we postpone the proof of Tutte's theorem, as well as of the results stated below, to the end of this section. In fact we prove the following equivalent version of Tutte's theorem.

Theorem 1.4.5
Let A be binary matrix. Then the following are equivalent:
(i) A has a totally unimodular signing.
(ii) A cannot be transformed to

$$M(F_7) := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

by applying (repeatedly) the following operations:
- deleting rows of columns;
- permuting rows or columns;
- taking the transposed matrix;
- pivoting over GF(2). □

A $\{0,\pm1\}$-matrix $\tilde{A}$ is called a *signing* of a binary matrix A if and only if $\tilde{A} \equiv A$ (modulo 2).

The link between Theorem 1.4.4 and Theorem 1.4.5 is the following theorem, due to Tutte, stating that in a sense "regular matroid" = "totally unimodular matrix".

<u>Theorem 1.4.6</u> (Tutte [1958])
Let $\mathcal{M}$ be a binary matroid with binary standard representation $[I|A]$. *Then $\mathcal{M}$ is regular if and only if* A *has a totally unimodular signing* $\tilde{A}$.
In that case $[I|\tilde{A}]$ *is a representation of $\mathcal{M}$ over* $\mathbb{R}$. □

A useful generalization of this theorem is:

<u>Theorem 1.4.7</u>
Let $\mathcal{M}$ be a binary matroid. Let $\mathcal{M}$ be a (not necessarily standard) representation of $\mathcal{M}$ over GF(2). *Then $\mathcal{M}$ is regular if and only if there exists a signing* N *of $\mathcal{M}$ representing $\mathcal{M}$ over* $\mathbb{R}$. *Moreover, each* $x \in \mathcal{N}(M)$ *as a signing* $y \in \mathcal{N}(N)$. □

<u>Remark</u>:

Let $\Gamma(\mathcal{M})$ be the matrix with rows all elements of $\mathcal{C}(\mathcal{M})$ ($\mathcal{M}$ binary). Then $\mathcal{M}$ is regular, if and only if, $\Gamma(\mathcal{M})$ and $\Gamma(\mathcal{M}^*)$ have a signing $\Sigma(\mathcal{M})$, $\Sigma(\mathcal{M}^*)$ respectively, such that $\Sigma(\mathcal{M})\Sigma(\mathcal{M}^*)^T = 0$. The latter property is called the *orientability* of a matroid (cf. Minty [1966]). "Only if" in the above equivalence easily follows by applying Theorem 1.4.7 to $M = \Gamma(\mathcal{M}^*)$.

OUTLINE

The remainder of this section is devoted to the proofs of the four theorems stated above. After some preliminaries on bipartite graphs, pivoting and total unimodularity we first prove Theorem 1.4.5. Next we prove Theorem 1.4.4 and Theorem 1.4.6 together. After proving two characterizations of totally unimodular matrices (Theorem 1.4.12), we prove Theorem 1.4.7. Finally we sketch a proof of Theorem 1.4.6 independent of Theorem 1.4.5.

THE BIPARTITE GRAPH OF A MATRIX

<u>Lemma 1.4.8</u>
Let G *be a connected simple bipartite graph. If deleting any pair of distinct nodes in the same colour-class yields a disconnected graph, then* G *is either a path or a circuit.*

Proof: Suppose G is neither a path nor a circuit. Then G has a spanning tree with at least three endpoints. At least two of these endpoints are in the same colour-class. Deleting these two nodes from G results in a connected graph. □

We apply this lemma to proving Theorem 1.4.5 on the bipartite graph of a matrix. Let A be a matrix (over any field). Denote the index-set of the rows (columns) of A by R(A) (C(A) respectively). The *bipartite graph*, G(A), *associated with* A has colour-classes R(A) and C(A). There is an edge from $r \in R(A)$ to $s \in C(A)$ if and only if the entry A_{rs} is non-zero.

PIVOTING

Let A be a matrix over a field **F**. Consider the pivot operation (1.4.2) ($\varepsilon \neq 0$):

$$A = \left[\begin{array}{c|c} \varepsilon & y^T \\ \hline x & D \end{array}\right] \longrightarrow B = \left[\begin{array}{c|c} -\varepsilon & y^T \\ \hline x & D-\varepsilon^{-1}xy^T \end{array}\right]$$

Then the following assertions hold:

(1.4.9) (i) pivoting B on $-\varepsilon$ yields A;

(ii) if A is square then $\det A = -\varepsilon \det(D-\varepsilon^{-1}xy^T)$;

(iii) if A is totally unimodular then B is totally unimodular;

(iv) if G(A) is connected then G(B) is connected.

[The proofs of (i), (ii) and (iii) are straightforward. To see (iv), consider that if G(B) is disconnected then G(A) is disconnected too.]

UNIQUENESS OF TOTALLY UNIMODULAR SIGNING

If A is a binary matrix that has a totally unimodular signing, then this signing is not unique (unless A is the all-zero matrix). Indeed, multiplying some rows and columns of a totally unimodular matrix by -1 yields a totally unimodular matrix again. Theorem 1.4.11 below states that this is the only freedom one has in making a totally modular signing of A. To prove this we need the following easy lemma.

Lemma 1.4.10
Let A *be a* $n \times n$*-matrix, with* $\{0, \pm 1\}$ *entries only. If* $G(A)$ *is a circuit, then* A *is totally unimodular if and only if the number of* -1*'s in* A *is congruent to* n *modulo* 2. □

Theorem 1.4.11 (Camion [1963])
Let M_1 *and* M_2 *be totally unimodular matrices, with* $M_1 \equiv M_2$ *(modulo 2). Then* M_1 *can be obtained from* M_2 *by multiplying some rows and columns of* M_2 *by* -1.

Proof: (Paul Seymour) Construct a signed graph (G,Σ) as follows: $G := G(M_1)$ $(=G(M_2))$. We call an edge in G even if the corresponding entries in M_1 and M_2 are the same. The other edges are odd (i.e, are in Σ). By Lemma 1.4.10 each chordless circuit in G is an even circuit in (G,Σ). Hence, so is any circuit. This means that the signed graph (G,Σ) is bipartite. Take $U \subseteq V(G)$ such that $\Sigma = \delta(U)$. Multiply by -1 all columns and rows of M_1 with index in U. This yields M_2. □

We are now able to prove Theorem 1.4.5.

Proof of Theorem 1.4.5 (Gerards [1987])

Let A be a binary matrix. The existence of a totally unimodular signing is invariant under the operations in Theorem 1.4.5 (ii) (by 1.4.9 (iii)). Moreover $M(F_7)$ has no totally unimodular signing. Hence (i) implies (ii). So it remains to prove the reverse implication.

Suppose A is a $\{0,1\}$-matrix, satisfying (ii), with no totally unimodular signing. We may assume that each proper submatrix of A has a totally unimodular signing. So the bipartite graph $G(A)$ is connected. (If not,

$$A = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right]$$

for certain matrices B and C (up to permutations of rows and columns), implying that at least one of B and C has no totally unimodular signing.)

G(A) is not a path or circuit (as otherwise A has trivially a totally unimodular signing). Hence, by Lemma 1.4.8, A or A^T is equal to $[x|y|N]$ (up to permutation of columns), where x and y are two column vectors and where G(N) is connected. By assumption, both $[x|N]$ and $[y|N]$ have a totally unimodular signing. Moreover, by Theorem 1.4.11, these two signings can be chosen so that in both cases N is signed in the same way. Hence A has a signing $A' = [x'|y'|N']$ satisfying:

(*) (i) G(N') is connected,

(ii) both $[x'|N']$ and $[y'|N']$ are totally unimodular.

Claim: *We may assume that matrix* $[x'|y']$ *has a submatrix of the form*

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Proof of the Claim: By (1.4.9) (iii) and (iv), pivoting A' on an entry in N' does not influence property (*). Now, pivot A' on an entry in N' such that the smallest submatrix M with determinant not equal to 0, 1, or -1, is as small as possible. Then M is a 2×2-matrix. (If not, pivot on an entry lying both in M and N', cf. (1.4.9) (ii)).) So M is of the form as in the claim (if necessary multiply x', y', or a row by -1). Moreover, by (*)(ii) M has to be a submatrix of $[x'|y']$.

end of proof of claim

Denote by α and β the row-indices of the two rows of A' in which the submatrix of the claim occurs. Since G(N') is connected there exists a path in G(N') from α to β. This path cannot have length 2 (as such a path would correspond to a column of N' with two ±1's in the rows α and β, contradicting the fact that both $[x'|N']$ and $[y'|N']$ are totally unimodular). From this it follows that A' has a submatrix of the form depicted in the figure below. (If necessary permute rows of A and columns of N', multiply them by -1, or exchange x' and y'.)

$$\begin{array}{c} \alpha \\ \beta \\ \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{c|c|cccccc} 1 & 1 & 1 & 0 & \cdots & & 0 & 0 \\ \hline 1 & -1 & 0 & 0 & \cdots & & 0 & 1 \\ \hline & & 1 & \underline{1} & & & & \\ & & & 1 & \underline{1} & & 0 & \\ * & * & & & \ddots & \ddots & & \\ & & 0 & & & \ddots & \underline{1} & \\ & & & & & & 1 & 1 \end{array}\right]$$

By pivoting on the underlined entries, deleting the rows and columns containing these pivot elements, and multiplying some rows and columns by -1 (and if necessary exchanging x' and y'), we get a submatrix of the form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ a & b & 1 & 1 \end{bmatrix}$$

It is still the case that deleting any of the first two columns yields a totally unimodular matrix. This implies that a = 1 and b = 0. Hence A can be transformed to $M(F_7)$, contradicting our assumption. □

Proof of Theorem 1.4.4 and Theorem 1.4.6

Both theorems follow from the following observations:

- Suppose a binary matrix A has a totally unimodular signing $\widetilde{A}$. Then a subdeterminant of A is nonzero, if and only if the corresponding subdeterminant of $\widetilde{A}$ is nonzero. This means that $[I|\widetilde{A}]$ is a real representation of the binary matroid represented over GF(2) by $[I|A]$.
- If $\mathcal{M}$ has an F_7 or F_7^* minor, then by Theorem 1.4.3 $\mathcal{M}$ is not regular (as F_7 is not regular).
- Taking a minor of a binary matroid represented by a binary matrix $[I|A]$, corresponds to deletion of rows and columns from A, combined with pivoting in A. Replacing the matroid by its dual corresponds to taking the transpose of A. □

In order to prove Theorem 1.4.7 we need the following characterization of totally unimodular matrices, due to Ghouila-Houri [1962] and Gomory (cf. Camion [1965]).

Theorem 1.4.12

Let A be a matrix with entries 0, 1 and -1. Then the following are equivalent.

(i) A *is totally unimodular;*

(ii) *For each* $\{0,1\}$*-matrix* B *there exists a signing* $\tilde{B}$ *such that* $\tilde{B}A$ *has* $\{0,\pm1\}$ *entries only* (Ghouila-Houri [1962]);

(iii) A *has no subdeterminant equal to* 2 *or* -2 (Gomory, (cf. Camion [1962])).

Proof:

(i) ⟹ (ii): Let A be totally unimodular. Let B be a $\{0,1\}$ matrix. We may assume that in fact B is a row-vector y^T.

Consider $P := \{x \mid 0 \leqq x \leqq y,\ \lfloor \frac{1}{2}y^TA \rfloor \leqq x^TA \leqq \lceil \frac{1}{2}y^TA \rceil\}$. As $\frac{1}{2}y \in P$, $P \neq \emptyset$. So, by Theorem 1.2.15 there exists an integer vector $x \in P$. Setting $\tilde{y} = y - 2x$ it is easy to see that $\tilde{y}^TA$ is a $\{0,\pm1\}$-vector.

(ii) ⟹ (iii): It suffices to show that if A is a square integral matrix with det A = ±2, there exists a B violating (ii). Therefore, let A be a square integral matrix, with det A = ±2. Then $2A^{-1}$ is an integral matrix (Cramer's rule). Let B be the $\{0,1\}$-matrix such that $B \equiv 2A^{-1}$ (modulo 2). Let $\tilde{B}$ be any signing of B. Then $\tilde{B}A = 2A^{-1}A = 2I \equiv 0$ (modulo 2). Suppose $\tilde{B}A$ has $\{0,\pm1\}$ entries only. Then $\tilde{B}A \equiv 0$, so as A is nonsingular, $\tilde{B} = 0$. Hence A^{-1} is integral ($2A^{-1} \equiv \tilde{B}$ (modulo 2)). However this contradicts $\det A^{-1} = \frac{1}{2}$.

(iii) ⟹ (i): It suffices to show that if A is a square $\{0,\pm1\}$-matrix, such that all proper subdeterminants of A are 0, 1 or -1, then $\det A \in \{0,\pm1,\pm2\}$.

Let A be a minimal counterexample to this. As all 2x2-matrices with $\{0,\pm1\}$ entries have determinant 0, 1, -1, 2, or -2 (as is easily checked), A has size at least 3. Now pivot A on some entry $A_{ij} \neq 0$ then delete row i and column j. Call the resulting matrix M. All proper subdeterminants of M are $\{0,\pm1\}$ (1.4.9(iii)). Moreover det M = ± det A. This contradicts the fact that A is a minimal counterexample. □

Using Theorem 1.4.12 we prove Theorem 1.4.7.

Proof of Theorem 1.4.7

Let $\mathcal{M}$ be a regular matroid, represented over GF(2) by A. Let A_{11} be a non-singular submatrix of A with $r(A_{11}) = r(A)$. We may assume that A has the following form

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right].$$

(So $A_{22} = A_{21}A_{11}^{-1}A_{12}$.) Then $\left[\begin{array}{c|c} I & A_{11}^{-1}A_{12}\end{array}\right]$ is a standard representation of $\mathcal{M}$ over GF(2). Let B be a totally unimodular signing of $A_{11}^{-1}A_{12}$ (Theorem 1.4.6). For i=1,2, let D_{i1} be a signing of A_{i1} such that $D_{i1}B$ is a matrix with entries 0, ± 1 only (cf. Theorem 1.4.12(ii)). Then the matrix

$$D = \left[\begin{array}{c|c} D_{11} & D_{11}B \\ \hline D_{21} & D_{21}B \end{array}\right]$$

is a signing of A. Moreover D represents $\mathcal{M}$ over $\mathbb{R}$, as D_{11} is nonsingular (as a real matrix). Indeed det $D_{11} \equiv$ det $A_{11} \neq 0$ (modulo 2), so det $D_{11} \neq 0$.

To prove the second part of Theorem 1.4.7 we may assume that M is a standard representation [I|D]. So M has a totally unimodular signing [I|B]. From this it is not hard to see that it suffices to show that each binary vector $\hat{x}$ has a signing $\hat{y}$ such that $B\hat{y}$ is a signing of $D\hat{x}$. This is an immediate consequence of Theorem 1.4.12 ((i)⇔(ii)). □

We conclude this section with a sketch of a proof of Theorem 1.4.6, which does not depend on Theorem 1.4.5. Let [I|B] be a real standard representation of a binary matroid $\mathcal{M}$ represented over GF(2) by a binary matrix [I|A]. It suffices to prove that we can multiply the rows and columns of B by nonzero reals such that we obtain a signing $\tilde{B}$ of A. Indeed, suppose we can, let $\tilde{B}$ be the resulting signing. Then $[I|\tilde{B}]$ is also a real representation of $\mathcal{M}$. So a subdeterminant of $\tilde{B}$ is nonzero if and only if the corresponding subdeterminant of A is nonzero. This means in particular that all even subdeterminants of $\tilde{B}$ are zero. So by Theorem 1.4.12 $\tilde{B}$ is totally unimodular.

To see that $\tilde{B}$ exists observe the following:

(i) Each subdeterminant of B is nonzero if and only if the correspond subdeterminant of A is nonzero.

(ii) $G(B) = G(A)$ (from (i)).

(iii) If $u_1v_1, v_1u_2, u_2v_2, v_2u_3, \ldots, u_kv_k, v_ku_1$ is a chordless circuit in $G(B)$, then the submatrix

$$\begin{bmatrix} B_{u_1v_1} & & & & B_{u_1v_k} \\ B_{u_2v_1} & B_{u_2v_2} & & 0 & \\ & B_{u_3v_2} & \ddots & & \\ & & & \ddots & \\ 0 & & & B_{u_kv_{k-1}} & B_{u_kv_k} \end{bmatrix}$$

(which exists in B) has determinant zero (by (ii) and (i)).

(iv) Let D be the directed graph obtained by replacing each $rc \in E(G(B))$ by $\overrightarrow{rc}$ and $\overrightarrow{cr}$ $(r \in R(B), c \in C(B))$. Define $w \in \mathbb{R}^{A(D)}$ by $w_{\overrightarrow{rc}} := -w_{\overrightarrow{cr}} :=$ $\log |B_{rc}|$ $(r \in R(B), c \in C(B))$.
From (iii) it follows that all directed circuits in D have length zero (for length function w).

(v) Let α_r $(r \in R(B))$, and β_c $(c \in C(B))$ be such that $\alpha_r + \beta_c =$ $w_{\overrightarrow{rc}}$ $(rc \in E(G(B)))$. (The numbers α_r, β_c exist by (iv) and Theorem 1.3.2.) By multiplying each row r of B by $e^{-\alpha_r}$ and each column c of B by $e^{-\beta_c}$ we get the desired matrix $\tilde{B}$.

CONCLUDING REMARKS

Above we gave an exposition of almost all basic notions of binary and regular matroids to be used in this monograph. The exception is Seymour's decomposition theorem for regular matroids (Seymour [1980]). We state this theorem in Section 3.2 (Theorem 3.2.1). Seymour's theorem says that graphic matroids are in a sense the only examples of regular matroids.

CHAPTER 2. CUTTING PLANES

A central problem in polyhedral combinatorics is the following:

> Given a polyhedron $\{x \in \mathbb{R}^n | Ax \leqq b\}$, find a system of linear inequalities $Mx \leqq d$ such that $\{x \in \mathbb{R}^n | Mx \leqq d\}$ is the convex hull of $\{x \in \mathbb{Z}^n | Ax \leqq b\}$.

In Section 2.1 we describe an iterative procedure (developed by Chvátal and Schrijver) for this problem. The number of iterations needed in this procedure is finite (Chvátal [1973], Schrijver [1980]). Moreover, it can be bounded from above by a function of A only (i.e., independently of b) (Cook, Gerards, Schrijver en Tardos [1986]). In Section 2.2, we give a short proof for that result. In the final section of this chapter, Section 3.3, we give a class of matrices A for which the number of iterations in the above mentioned procedure is at most 1.

2.1. CUTTING PLANES - FINDING THE INTEGER HULL OF A POLYHEDRON

From Meyer's Theorem (Theorem 1.2.14) we know that the integer hull P_I of a rational polyhedron P in $\mathbb{R}^n$ is a polyhedron too. So $P_I = \{x \in \mathbb{R}^n | Mx \leqq d\}$ for some matrix M and vector d. We now describe a finite procedure to find M and d, which is developed by Chvátal [1973] and Schrijver [1980].

Let $\mathcal{H}_P$ be the set of all rational halfspaces containing P. We define the *Chvátal closure* P' of P by

$$(2.1.1) \quad P' = \bigcap_{H \in \mathcal{H}_P} H_I.$$

Remark:

Let H be a rational halfspace in $\mathbb{R}^n$. Then, clearly there exists an $a \in \mathbb{Z}^n$, and $\alpha \in \mathbb{R}$ such that the greatest common divisor of the components of vector a is equal to 1. In that case $H_I = \{x \in \mathbb{R}^n | a^T x \leqq \lfloor \alpha \rfloor\}$, as is easily verified.

Obviously the convex set P' satisfies

$$(2.1.2) \quad P_I \subset P' \subset P.$$

The following result shows that P' is a "better" approximation of P_I than P itself (unless $P = P_I$).

Theorem 2.1.3

Let P *be a rational polyhedron in* $\mathbb{R}^n$. *Then the following hold:*

(i) $P' = P$ *if and only if* $P = P_I$;

(ii) P' *is a rational polyhedron.*

Proof: Let $Ax \leqq b$ be a totally dual integral system defining P, with $A \in \mathbb{Z}^{m \times n}$. (cf. Theorem 1.2.21). Then $P' = \{x \in \mathbb{R}^n | Ax \leqq \lfloor b \rfloor\}$. Indeed, it is obvious that $P' \subset \{x \in \mathbb{R}^n | Ax \leqq \lfloor b \rfloor\}$. Conversely, if $a^T x \leqq \alpha$ is valid for P with $a \in \mathbb{Z}^n$ then $a = y^T A$, $\alpha \geqq y^T b$ for some $y \in \mathbb{Z}^m_+$. Hence $a^T x \leqq \lfloor \alpha \rfloor$ is valid for

$\{x \in \mathbb{R}^n | Ax \leqq \lfloor b \rfloor\}$ as $Ax \leqq \lfloor b \rfloor$ implies $a^T x = y^T Ax \leqq y^T \lfloor b \rfloor \leqq \lfloor y^T b \rfloor \leqq \lfloor \alpha \rfloor$. So P' is a rational polyhedron. Moreover, by Theorem 1.2.17, $P' = P$ if and only if $P = P_I$. □

The best that can happen in case $P \neq P_I$ is that $P' = P_I$. However this is generally not the case.

Example 2.1.4

Let $P(a) := \{[x_1, x_2]^T \in \mathbb{R}^2 | 2ax_1 - x_2 \geqq 0,\ 2ax_1 + x_2 \leqq 2a,\ x_2 \leqq 0\}$ for $a \in \mathbb{N}$. Then $P(a)_I = \{[x_1, x_2]^T \in \mathbb{R}^2 | 0 \leqq x_1 \leqq 1,\ x_2 = 0\}$ $(=: P(0))$. However $P'(a) \supset P(a-1) \neq P_I(a)$ for $a \geqq 1$.

We define the following sequence of polyhedra:

$$P^{(0)} := P;$$
$$(2.1.5) \qquad P^{(i)} := (P^{(i-1)})' \quad \text{if} \quad i = 1,2,\ldots .$$

From Theorem 2.1.3 if follows that:

$$P = P^{(0)} \supset P^{(1)} \supset \ldots \supset P^{(i)} \supset P^{(i+1)} \supset \ldots \supset P_I;$$
$$(2.1.6) \qquad P^{(i)} = P^{(i+1)} \text{ if and only if } P^{(i)} = P_I.$$

Moreover we have:

Theorem 2.1.7 (Chvátal [1973]. Schrijver [1980])

Let P *be a rational polyhedron in* $\mathbb{R}^n$. *Then there exists a* $t \in \mathbb{N}$ *such that* $P^{(t)} = P_I$. □

We call the smallest t such that $P^{(t)} = P_I$ the *Chvátal rank* of P.

So we can iteratively determine systems of linear inequalities describing $P^{(1)}, P^{(2)}, \ldots$. After a finite number of iterations one has $P^{(i)} = P^{(i+1)}$ (which can be checked using linear programming methods), which means that the system describing $P^{(i)}$, describes P_I. That this procedure can be carried out in a finite number of steps follows from the following lemma.

Lemma 2.1.8

Let $A \in \mathbb{Q}^{m \times n}$ *and let* $b \in \mathbb{Q}^m$. *Then there exists a finite algorithm that determines a matrix* M_1 *and a vector* d_1 *such that*

$$\{x \in \mathbb{R}^n \mid Ax \leqq b\}' = \{x \in \mathbb{R}^n \mid M_1 x \leqq d_1\}.$$

Proof: Let Y denote the set of all $y \in \mathbb{R}^m$, such that $y^T A \in \mathbb{Z}^n$, $0 \leqq y \leqq 1$, and such that the rows a_i^T of A with $y_i \neq 0$ are linearly independent. It is easy to see, by Cramer's rule, that y is a finite set, which can be determined by a finite algorithm. Moreover, the system $(y^T A)x \leqq y^T b$, $y \in Y$ is a totally dual integral system for $\{x \in \mathbb{R}^n \mid Ax \leqq b\}$ (= : P). Hence P' = $\{x \in \mathbb{R}^n \mid (y^T A)x \leqq \lfloor y^T b \rfloor,\ y \in Y\}$. □

The procedure indicated by (2.1.6) can be viewed as a polyhedral version of Gomory's cutting plane method for integer linear programming (Gomory [1958, 1960, 1963]).

It should be noted that Theorem 2.1.7 and Lemma 2.1.8 do not give a polynomial-time algorithm to find a description of P_I in terms of linear inequalities. There are two reasons for this. The first reason is that the number of facets of P' can be exponential in the size of the description of P. (see, for example, (1.2.11); this system describes P' in case P is the polyhedron described by the inequalities in the first two lines of (1.2.11)). Secondly, the Chvátal rank of a polyhedron can be exponential too. Indeed, the input size of P(a) in Example 2.1.4 is O(log(a)). The Chvátal rank of P(a) is at least a.

On the other hand, there is some indication that solving max $\{c^T x \mid x \in P \cap \mathbb{Z}^n\}$ (which is $\mathcal{NP}$-hard, Cook [1971]), is not so hard in case P has low Chvátal-rank. To see this observe the following two facts.

(2.1.9) Let $x_0 \in P \cap \mathbb{Z}^n$. Then in case x_0 is not an optimal solution of $\max\{c^T x \mid x \in P \cap \mathbb{Z}^n\}$ this can be proved in polynomial-time by giving a better feasible solution: $y_0 \in P \cap \mathbb{Z}^n$, $c^T y_0 > c^T x_0$. (The fact there exists an y_0 of polynomial size follows from the remark following Theorem 2.2.2.)

(2.1.10) Let $x_0 \in P'$. Then, in case x_0 is an optimal solution of $\max\{c^T x \mid x \in P'\}$, optimality of x_0 can be proved in polynomial-time by giving integer vectors m_i, rationals β_i, and non-negative numbers γ_i $(i=1,\dots,n)$ such that:

- $x \in P \Rightarrow m_i^T x \leqq \beta_i$ (this can be proved using a polynomial-time linear programming algorithm);
- $c^T = \sum_{i=1}^{n} \gamma_i m_i^T$;
- $c^T x_0 = \sum_{i=1}^{n} \gamma_i \lfloor \beta_i \rfloor$.

(If $P = \{x \in \mathbb{R}^n \mid Ax \leqq b\}$, then using a description for P' as given in the proof of Lemma 2.1.8, one can prove that the sizes of m_i, γ_i and β_i can be taken polynomial in the size of A and c).
This means that integer programming over polyhedra with Chvátal rank 1 has a good characterization. (For the case that P is not given by a system of inequalities but by a separation algorithm, and for generalization to higher Chvátal rank see Boyd and Pulleyblank [1984], and Schrijver [1986, Section 23.6]).

COMPUTATIONAL COMPLEXITY OF THE CHVATAL RANK

How hard is it to decide the Chvátal rank of a polyhedron? The answer to this question is open. Even the decision problem "Given a matrix $A \in \mathbb{Z}^{m\times n}$ and a vector $b \in \mathbb{Z}^m$, has the polyhedron $\{x \in \mathbb{R}^n \mid Ax \leqq b\}$ Chvátal rank 0 (i.e. is the polyhedron integral)?" is only known to be in co-$\mathcal{NP}$. It is open whether this problem is $\mathcal{NP}$-complete, well-characterized or (and) in $\mathcal{P}$. (The fact that it is in co-$\mathcal{NP}$ is easy to prove.)

On the other hand the decision problems "Given a matrix $A \in \mathbb{Z}^{m\times n}$ has $\{x \in \mathbb{R}^n \mid Ax \leqq b\}$ Chvátal rank 0 for each $b \in \mathbb{Z}^m$?" and "Given a matrix $A \in \mathbb{Z}^{m\times n}$ has $\{x \in \mathbb{R}^n \mid d_1 \leqq x \leqq d_2;\ b_1 \leqq Ax \leqq b_2\}$ Chvátal rank 0 for each $d_1, d_2 \in \mathbb{Z}^n$; $b_1, b_2 \in \mathbb{Z}^m$?" both are in $\mathcal{P}$. [Indeed, from Theorem 1.2.16, Theorem 1.2.15 respectively, it follows that solving these problems amounts to decide wether or not A^T is unimodular, A is totally unimodular respectively. The fact that these problems can be solved in polynomial time follows essentially from Seymour's decomposition theorem for regular matroids (Seymour [1980], cf. Theorem 3.2.1). For details and references we refer to Schrijver [1986, Section 19.4, Chapter 20, and Theorem 21.6 (due to Truemper [1978])].]

This motivates the question: "Given a matrix $A \in \mathbb{Z}^{m \times n}$ has $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ Chvátal rank at most t for each $b \in \mathbb{Z}^m$?". Unfortunately, in general not much is known on this question. However, for each matrix A there exists a $t \in \mathbb{N}$ such that the answer to the question becomes "yes" (Section 2.2). Moreover, in Section 2.3 we consider two classes of matrices A such that $\{x \mid d_1 \leq x \leq d_2;\ b_1 \leq Ax \leq b_2\}$ has Chvátal rank at most 1 for each integral d_1, d_2, b_1 and b_2. One of these classes is due to Edmonds and Johnson [1970], the other class is due to Gerards and Schrijver [1986]. In both cases membership-testing for these classes is in $\mathcal{P}$.

2.2. THE CHVATAL RANK OF A MATRIX

In this section we prove

Theorem 2.2.1 (Cook, Gerards, Schrijver and Tardos [1986])
Let $A \in \mathbb{Z}^{m \times n}$. *Then there exists a* $t \in \mathbb{N}$ *such that*

$$\{x \in \mathbb{R}^n \mid Ax \leqq b\}_I = \{x \in \mathbb{R}^n \mid Ax \leqq b\}^{(t)}$$

for each $b \in \mathbb{Z}^n$. □

In fact, Cook, Gerards, Schrijver and Tardos prove that the number t in Theorem 2.2.1 can be taken equal to $2^{n^3+1} n^{5n} \Delta(A)^{n+1}$. ($\Delta(A)$ denotes the largest absolute value of a subdeterminant of A.) The proof of this result which we give below does not yield this explicit value. The result of Theorem 2.2.1, which is implicitly proved earlier by Blair and Jeroslow [1982] (cf. Cook, Gerards, Schrijver, and Tardos [1986]), makes the following definition meaningful. Let $A \in \mathbb{Z}^{m \times n}$. Then the *Chvátal rank* of A is the smallest integer t such that $\{x \in \mathbb{R}^n \mid Ax \leqq b\}_I = \{x \in \mathbb{R}^n \mid Ax \leqq b\}^{(t)}$ for all $b \in \mathbb{Z}^m$. The *strong Chvátal rank* is the Chvátal rank of

$$\begin{bmatrix} I \\ -I \\ A \\ -A \end{bmatrix}$$

The proof of Theorem 2.2.1 makes use of the following result.

Theorem 2.2.2
Let P *be a rational polyhedron in* $\mathbb{R}^n$. *Then there exists a finite set* L *in* $\mathbb{Z}^n$ *such that for each* $w \in \mathbb{Z}^n$ *and* $\tilde{z} \in P \cap \mathbb{Z}^n$ *one of the following holds*
(i) $w^T \tilde{z} = \max\{w^T x \mid x \in P \cap \mathbb{Z}^n\}$;
(ii) there exists a $z' \in L$ *such that* $w^T \tilde{z} < w^T z'$ *and* $\tilde{z} + z' \in P$. □

[Theorem 2.2.2 easily follows from Meyer's theorem (Theorem 1.2.14) and Motzkin's theorem (Theorem 1.2.2). In case $P = \{x\in\mathbb{R}^n | Ax \leqq b\}$, where A is an integral matrix, L can be chosen independently of vector b (Graver [1975], cf. Blair and Jeroslow [1982]). In fact we can set $L = \{x\in\mathbb{Z}^n \,\big|\, |z_i| < n\Delta(A)\ (i=1,\dots,n)\}$ (Cook, Gerards, Schrijver and Tardos [1982]).]

Proof of Theorem 2.2.1

Suppose the result is not true. Then there exists a matrix $A\in\mathbb{Z}^{m\times n}$ and sequences $\{b_k\}_{k\in\mathbb{N}}$, $\{w_k\}_{k\in\mathbb{N}}$ and $\{\beta_k\}_{k\in\mathbb{N}}$ in $\mathbb{Z}^m$, $\mathbb{Z}^n$ and $\mathbb{Z}$ respectively such that for each $k\in\mathbb{N}$:

(i) $w_k^T x \leqq \beta_k$ is valid for $\{x\in\mathbb{R}^n | Ax \leqq b_k\}_I$;

(ii) $w_k^T x \leqq \beta_k$ is not valid for $\{x\in\mathbb{R}^n | Ax \leqq b_k\}^{(k)}$.

Obviously (by taking subsequences, if necessary) we may assume that $\{x\in\mathbb{Z}^n | Ax \leqq b_k\}$ is empty for each k or is not empty for each k. So we have two cases:

Case I: $\{x\in\mathbb{Z}^n | Ax \leqq b_k\} = \emptyset$ *for each* $k\in\mathbb{N}$.

We may assume that $\{x\in\mathbb{R}^n | Ax \leqq b_k\} \neq \emptyset$ for each $k\in\mathbb{N}$. (Indeed, by Theorem 2.1.7 and assumptions (i) and (ii) above only a finite number of $\{x\in\mathbb{R}^n | Ax \leqq b_k\}$ can be empty. So, by taking a subsequence, we may assume that none of these polyhedra is empty.) Let $x_k\in\mathbb{R}^n$ with $Ax_k \leqq b_k$. By replacing b_k by $b_k - A\lfloor x_k \rfloor$ we may assume that $0 \leqq (x_k)_i \leqq 1$ for each $k\in\mathbb{N}$, $i=1,\dots,m$. Hence we may assume that for each $i=1,\dots,n$ the sequence $\{b_k\}_{k\in\mathbb{N}}$ is, componentwise, bounded from below. Split the system $Ax \leqq b_k$ into two (possibly empty) subsystems $Cx \leqq c_k$ and $Dx \leqq d_k$ such that $\{c_k\}_{k\in\mathbb{N}}$ is bounded, and $\{d_k\}_{k\in\mathbb{N}}$ is componentwise unbounded. By taking subsequences again, we may assume that $c_k = c$ for all $k\in\mathbb{N}$ and some fixed c and that $\{d_k\}_{k\in\mathbb{N}}$ tends componentwise to infinity. Hence $\{x\in\mathbb{Z}^n | Cx \leqq c\} = \emptyset$ (if not $\{x\in\mathbb{Z}^n | Ax \leqq b_k\} \neq \emptyset$ for some $k\in\mathbb{N}$). Let t be the Chvátal rank of $\{x\in\mathbb{Q}^n | Cx \leqq c\}$ (Theorem 2.1.7). Then $w_t x \leqq \beta_t$ is valid for $\{x\in\mathbb{R}^n | Cx \leqq c\}_I = \{x\in\mathbb{R}^n | Cx \leqq c\}^{(t)} \supset \{x\in\mathbb{R}^n | Ax \leqq b_t\}^{(t)}$. Which contradicts assumption (ii) above.

<u>Case II</u>: $\{x \in \mathbb{Z}^n | Ax \leqq b_k\} \neq \emptyset$ *for each* $k \in \mathbb{N}$.

Without no loss of generality we may assume that $0 \in \{x \in \mathbb{Z}^n | Ax \leqq b_k\}$ for each $k \in \mathbb{N}$. As in Case I we may assume that we can split $Ax \leqq b_k$ into two systems $Cx \leqq c_k$ and $Dx \leqq d_k$, such that $c_k = c$ for all $k \in \mathbb{N}$ and some fixed c, and that $\{d_k\}_{k \in \mathbb{N}}$ tends componentwise to infinity. Let t be the Chvátal rank of $\{x \in \mathbb{Q}^n | Cx \leqq c\}$ (Theorem 2.1.7). Then by assumptions (i) and (ii) above $w_k^T x \leqq \beta_k$ is invalid for $\{x \in \mathbb{Q}^n | Cx \leqq c\}_I$ for each $k \geqq t$ (as $\{x \in \mathbb{Q}^n | Cx \leqq c\}^{(t)} \supset \{x \in \mathbb{Q}^n | Ax \leqq b_k\}^{(t)}$). Let $x_k \in \{x \in \mathbb{Z}^n | Cx \leqq c\}$ with $w_k^T x_k > \beta_k$, and $x_k \in L$ for $k \geqq t$, where L is the finite set of Theorem 2.2.2 for polyhedron $\{x \in \mathbb{Z}^n | Cx \leqq c\}$. As L is finite and $\{b_k\}_{k \in \mathbb{N}}$ tends componentwise to infinity there exists a $K \in \mathbb{N}$ such that $x_k \in \{x \in \mathbb{Z}^n | Ax \leqq b_k\}$ for $k > K$, which contradicts assumption (i) above.

Conclusion: Both in Case I and in Case II we derived a contradiction, which proves Theorem 2.2.1. □

2.3. MATRICES WITH THE EDMONDS-JOHNSON PROPERTY

We say that a matrix $A \in \mathbb{Z}^{m \times n}$ has the *Edmonds-Johnson property*, if it has strong Chvátal rank at most 1. Edmonds and Johnson [1970, 1973] derived from Edmonds' characterization of the matching polytope (Edmonds [1965c], cf. (1.2.11)) that if $A \in \mathbb{Z}^{m \times n}$ such that

$$(2.3.1) \qquad \sum_{i=1}^{m} |A_{ij}| \leq 2 \qquad (j=1,\ldots,n),$$

then A has strong Chvátal rank at most 1, and hence has the Edmonds-Johnson property.

The Edmonds-Johnson property is not maintained when passing to transposes; i.e. (2.3.1) may not be replaced by

$$(2.3.2) \qquad \sum_{j=1}^{n} |A_{ij}| \leq 2 \qquad (i=1,\ldots,n),$$

as the matrix

$$M(K_4) := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(the edge-node incidence matrix of the undirected graph K_4) does not have the Edmonds-Johnson property. Consider $0 \leq x \leq 1$; $0 \leq M(K_4)x \leq 1$.) In this section we show that, $M(K_4)$ is essentially the only counterexample among the matrices satisfying (2.3.2).

Theorem 2.3.3 (Gerards and Schrijver [1986])
Let $A \in \mathbb{Z}^{m \times n}$, *satisfying (2.3.2). Then* A *has the Edmonds-Johnson property if and only if* A *cannot be transformed to* $M(K_4)$ *by a series of the following operations:*

(2.3.4) *(i)* *deleting or permuting rows and columns, or multiplying then by -1;*

(ii) *replacing the matrix*

$$\left[\begin{array}{c|c} 1 & g^{\mathsf{T}} \\ \hline f & D \end{array}\right],$$

by the matrix

$$[D - fg^{\mathsf{T}}].$$ □

The operation (2.3.4)(ii) is called *contraction* (compare with the subsection **MINORS** of Section 1.4). The proof of Theorem 2.3.3 is at the end of this section. In this proof we make use of the fact that the Gomory cuts which essentially are needed to describe P' if $P = \{x \in \mathbb{R}^n | a_1 \leqq x \leqq a_2,\ b_1 \leqq Ax \leqq b_2\}$ such that $A \in \mathbb{Z}^{m \times n}$ satisfies (2.3.2), and a_1, $b_2 \in \mathbb{Z}^n$; $b_1, b_2 \in \mathbb{Z}^m$, are of a specific type. To describe this type of Gomory cuts we use the terminology of graph theory.

Any integral matrix A satisfying (2.3.2) can be considered as a *bidirected graph:* the columns of A correspond to the nodes of this graph, and the rows to the edges. A row containing two +1's corresponds to a ++ edge, connecting the two nodes (columns) where the +1's occur. Similarly, there are +- edges and -- edges. Moreover, there are ++ loops (if a 2 occurs) and -- loops (if a -2 occurs), (and + loops and - loops for rows with exactly one ± 1, but they will be irrelevant in this discussion). It will be convenient to identify the matrix with this bidirected graph. We denote the set of nodes (= columns) of a bidirected graph A by V(A) and the set of edges (=rows) by E(A).

A *circuit* in a bidirected graph is a square submatrix C of the form

$$\begin{bmatrix} \pm1 & \pm1 & & & & \\ & \pm1 & \pm1 & & 0 & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & 0 & & & \cdot & \cdot \\ & & & & \pm1 & \pm1 \\ \pm1 & & & & 0 & \pm1 \end{bmatrix} \text{ or } [\pm 2]$$

(possibly with rows or columns permuted).

Associated with a bidirected graph A, we define a signed graph $\Sigma(A)$ by considering ++ edges, -- edges, ++ loops, and -- loops as *odd* edges (loops respectively), and +- edges as *even* edges. In doing so we can use the terminology and results from, signed graphs for bidirected graphs. In particular a circuit in a bidirected graph is *odd* of it contains an odd number of ++ edges and -- edges.

So a matrix A satisfying (2.3.2) can not only be considered as a matrix, and a bidirected graph, but also yields a signed graph. Throughout this section we shall intertwine the terminoly of matrices, bidirected graphs and signed graphs. For example: a bidirected graph is bipartite if and only if it is totally unimodular, as is well-known and easy to prove.

If A is a bidirected graph, $x \in \mathbb{R}^{V(A)}$, $b \in \mathbb{Z}^{E(A)}$ we denote:

(2.3.5) $x(e)$:= entry in position e of Ax (so $x(e) = \pm x_u \pm x_v$ if e connects u and v).

So $Ax \leqq b$ is equivalent to: $x(e) \leqq b_e$ for $e \in E(A)$. If C is an odd circuit in A, the corresponding *odd circuit inequality* is, by definition:

(2.3.6) $$\tfrac{1}{2} \sum_{e \in E(C)} x(e) \leqq \left\lfloor \tfrac{1}{2} \sum_{e \in E(C)} b_e \right\rfloor$$

So, it is a special type of Gomory cut for the polyhedron $\{x\in\mathbb{R}^{V(A)} | Ax \leqq b\}$. In fact, for bidirected graphs, the odd circuit inequalities imply all Gomory cuts:

Lemma 2.3.7
Let A *be a bidirected graph, and let* $b\in\mathbb{Z}^{E(A)}$. *Then the system*

$$(2.3.8) \qquad \begin{array}{l} Ax \leqq b \\ c^T x \leqq \lfloor \delta \rfloor \quad (\textit{if } c\in\mathbb{Z}^{V(A)}, \textit{ and } Ax \leqq b \textit{ implies } c^T x \leqq \delta), \end{array}$$

has the same solution set as the system

$$(2.3.9) \qquad \begin{array}{l} Ax \leqq b \\ \frac{1}{2} \sum_{e\in E(C)} x(e) \leqq \lfloor \frac{1}{2} \sum_{e\in E(C)} b_e \rfloor \quad (C \textit{ odd circuit}) \end{array}$$

Proof: It suffices to show that each solution of (2.3.9) satisfies each inequality $c^T x \leqq \lfloor\delta\rfloor$ of (2.3.8). Choose $c\in\mathbb{Z}^{V(A)}$ such that $Ax \leqq b$ implies $c^T x \leqq \delta$. By the linear programming duality Theorem 1.2.6, (or by Farkas' Lemma (1.1.1)), $y^T A = c^T$, $y^T b \leqq \delta$ for some $y\in\mathbb{R}_+^{E(A)}$. By Carathéodory's Theorem, we may assume that the positive components of y correspond to linearly independent rows of A. As each non-singular submatrix of A has half-integral inverse (as is easily checked), it follows that y is half-integral (i.e. $2y\in\mathbb{Z}^{E(A)}$). Let A' be the submatrix of A consisting of those rows of A which have positive component in y. We consider two cases

Case I: A' *contains an odd circuit* C *(say).*

Let $\tilde{y} := \frac{1}{2}\chi_{E(C)}$, and let $\hat{y} := y-\tilde{y} \geqq 0$. If $\hat{y} = 0$, we know that

$$c^T x = \frac{1}{2} \sum_{e\in E(C)} x(e) \leqq \lfloor \frac{1}{2} \sum_{e\in E(C)} b_e \rfloor = \lfloor y^T b \rfloor \leqq \lfloor \delta \rfloor .$$

If $\hat{y}\neq 0$, applying induction on $\sum_{e\in E(A)} |y_e|$, we know that $(\hat{y}^T A)x \leqq \lfloor \hat{y}^T b \rfloor$ follows from (2.3.8). Hence:

$$c^T x = (y^T A)x = (\tilde{y}^T A)x + (\hat{y}^T A)x \leqq \lfloor \tilde{y}^T b \rfloor + \lfloor \hat{y}^T b \rfloor \leqq \lfloor y^T b \rfloor \leqq \lfloor \delta \rfloor .$$

Case II: A' *is bipartite.*

Then A' is totally unimodular, and hence (Theorem 1.2.16) $Ax \leqq b$ implies $c^T x = (y^T A)x \leqq \lfloor y^T b \rfloor \leqq \lfloor \delta \rfloor$. □

There is a strong relation between the operations (2.3.4) and reductions of $\Sigma(A)$. Deletion of rows of A means deletion of edges of $\Sigma(A)$. Deletion of columns of A means deletion of the corresponding node and the edges incident with it from A. Multiplying a column of A by -1 means resigning $\Sigma(A)$ on the corresponding node. The other operations in (2.3.4)(i) do not change $\Sigma(A)$.

What means contraction (operation 2.3.4(ii))? If we apply operation (2.3.4)(ii) and the first row in the initial matrix is a +- edge, we get the contraction of an even edge in $\Sigma(A)$. If the first row is a ++ edge or a -- edges, then operation (2.3.4)(ii) means resigning on the first node (to make the first edge (row) even) followed by the contraction of the, now even, edge in $\Sigma(A)$.

Thus we obtain the following equivalent form of Theorem 2.3.3 (cf. Lemma 1.3.3)

Corollary 2.3.10
A bidirected graph A *has the Edmonds-Johnson property if and only if* $\Sigma(A)$ *does not contain an odd-*K_4. □

A consequence of Corollary 2.3.10 is the following. An undirected graph G is called *t-perfect* if the convex hull of the characteristic vectors of stable sets in G is defined by:

$$\begin{array}{llll}
(2.3.11) & x_v & \geqq 0 & (v \in V); \\
& x_v + x_w & \leqq 1 & (vw \in E); \\
& \sum_{v \in V(C)} x_v & \leqq \lfloor \tfrac{1}{2} |V(C)| \rfloor & (C \text{ odd circuit in } G).
\end{array}$$

(A *stable set* in G, is a collection of mutually non-adjacent nodes in G.) Then Corollary 2.3.10, together with Lemma 2.3.7, directly gives

Corollary 2.3.12

If (G,E(G)) *contains no odd-*K_4*, then* G *is t-perfect.* □

This extends results of Chvátal [1975], Boulala and Uhry [1979], Sbihi and Uhry [1984], and Fonlupt and Uhry [1982] (see Section 3.6 for a discussion). There exist however t-perfect graphs which do not have the Edmonds-Johnson property, like the graph in Figure 2.1.

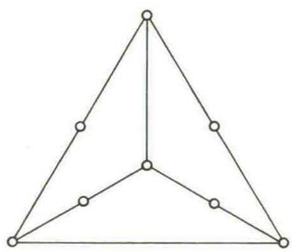


Figure 2.1

In Section 3.6 we shall extend Corollary 2.3.12 by proving that (2.3.11) is totally dual integral for graphs with no odd-K_4. Here we use structural properties of signed graphs with no odd-K_4, which are derived in Sections 3.1, 3.2, and 3.3.

Remarks:

(i) It follows with the ellipsoid method that if A is a bidirected graph with the Edmonds-Johnson property, and $b\in\mathbb{Q}^{E(A)}$ and $w\in\mathbb{Q}^{V(A)}$, we can solve the integer linear programming problem

$$\max\{w^{T}x \mid Ax \leqq b,\ x\in\mathbb{Z}^{V(A)}\} \tag{2.3.12}$$

in polynomial-time. By the results described by Grötschel, Lovász, and Schrijver [1981], to show polynomial solvability of (2.3.11) it suffices to give a polynomial-time separation algorithm for the convex hull of the solution set of (2.3.11). So we need a polynomial-time algorithm for the following problem

(2.3.13) Given a bidirected graph A and $z \in \mathbb{R}^{V(A)}$. Decide whether or not,

$$Az \leqq b$$

$$\tfrac{1}{2} \sum_{e \in E(C)} z(e) \leqq \lfloor \tfrac{1}{2} \sum_{e \in E(C)} b_e \rfloor \qquad \text{(C odd circuit)}$$

and, if not, find a violated inequality.

We here describe such an algorithm. First check $Az \leqq b$, if one of the constraints is violated, then we are done. Otherwise we must check the odd circuit constraints. It is not hard to see that for z satisfying $Az \leqq b$ the following two systems are equivalent

$$\tfrac{1}{2} \sum_{e \in E(C)} z_e \leqq \lfloor \tfrac{1}{2} \sum_{e \in E(C)} b_e \rfloor \qquad \text{(C odd circuit),} \tag{2.3.14}$$

$$\sum_{e \in E(C)} \ell_e \geqq 1 \qquad \text{(C circuit, } \sum_{e \in E(C)} b_e \text{ is odd),} \tag{2.3.15}$$

where ($\ell \in \mathbb{R}_+^{E(A)}$ is defined by $\ell := b - Az$).

[Indeed, (2.3.15) is equivalent to

$$\tfrac{1}{2} \sum_{e \in E(C)} z(e) \leqq \tfrac{1}{2} \sum_{e \in E(C)} b_e - 1 \qquad \text{(C circuit, } \sum_{e \in E(C)} b_e \text{ is odd).}$$

Moreover, if $Az \leqq b$, then $\tfrac{1}{2} \sum_{e E(C)} z_e \leqq \lfloor \tfrac{1}{2} \sum_{e \in E(C)} b_e \rfloor$ as soon as C is an even circuit, or $\sum_{e \in E(C)} b_e$ is even. So we see that (2.3.14) and (2.3.15) are equivalent, in case $Az \leqq b$.]

To check (2.3.15), split each node u in A(V) into two nodes u_+ and u_-, and make edges as:

(2.3.16) if $e \in E(A)$, connects u and v and b_e is even, make edges u_+w_+ and u_-w_-, each with length ℓ_e.
if $e \in E(A)$, connects u and v and b_e is odd, make edges u_+w_- and u_-w_+, each with length ℓ_e.

Then circuits C in A with b(C) odd correspond to paths from u_+ to u_- for some u. So finding a circuit C with b(C) odd and violating (2.3.14) is equivalent to finding a path from u_+ to u_-, of length less then 1, for some u. This can be done in polynomial-time, with a shortest path algorithm.

(ii) Using the remark (i) above one can prove that the decision problem "Given a bidirected graph A, has A the Edmonds-Johnson property?" is in co-$\mathcal{NP}$. A fact which also follows from Theorem 2.3.3 (Corollary 2.3.10). In fact the decision problem is in $\mathcal{P}$. However this does not follow immediately from Theorem 2.3.3. (In itself this theorem does not even give a good characterization for recognizing bidirected graphs with the Edmonds-Johnson property.) Truemper [1987] showed that for a binary matroid $\mathcal{M}$, and an element x of $\mathcal{M}$, it can be tested in polynomial-time whether or not $\mathcal{M}$ has an F_7-minor using x. This implies (cf. Theorem 3.1.2 (i)) that a bidirected graph can be tested in polynomial-time for having the Edmonds-Johnson property. The existence of a polynomial-time algorithm also follows from Theorem 2.3.3 together with Theorem 3.2.4. The latter theorem is a special instance of a result of Truemper and Tseng [1986].

(iii) There are three equivalent properties for a bidirected graph A:
a) A has the Edmonds-Johnson property;
b) $\Sigma(A)$ contains no odd-K_4;
c) The system:

$$\begin{aligned} x_e &\geq 0 && (e \in E(A)); \\ \sum_{e \in E(C)} x_e &\geq 1 && (C \text{ odd circuit}) \end{aligned}$$

is totally dual integral. (cf. Theorem 3.4.2).

Properties a) and c) are very much related, but we were not able to find a direct way of deriving one from the other. In fact, if the list of "minor-minimal counterexamples" for the "weak max-flow-min-cut"-property given by Seymour [1977, p. 200] is complete -which is not known-, then Theorem 2.3.3 would follow as a corollary.

If A has the Edmonds-Johnson property, and the polyhedron

$$(2.3.17)\quad \{x\in\mathbb{R}^{V(A)} \mid d_1 \leqq x \leqq d_2,\ b_1 \leqq Ax \leqq b_2\}$$

has Chvátal rank 0 or 1 for each integral d_1, d_2, b_1, and b_2, then (2.3.16) also has Chvátal rank 0 or 1 if some of the components in d_1, d_2, b_1 and b_2 are $\pm\infty$. This follows from the following lemma.

<u>Lemma 2.3.18</u>

Let $A\in\mathbb{Z}^{m\times n}$, $B\in\mathbb{Z}^{k\times n}$, *and* $b\in\mathbb{Z}^m$. *Moreover let* $b^1, b^2, \ldots \in\mathbb{Z}^k$ *such that* $b^1_i, b^2_i, \ldots$ *tends to infinity for each* $i=1,\ldots,k$.
If $P_j := \{x\in\mathbb{R}^n \mid Ax \leqq b,\ Bx \leqq b^j\}$ *has Chvátal rank* t *for each* $j\in\mathbb{N}$, *then* $P := \{x\in\mathbb{R}^n \mid Ax \leqq b\}$ *has Chvátal rank at most* t.

<u>Proof</u>: P'_j is constrained by the system:

$$(*)\quad \begin{cases} Ax \leqq b; \\ y^\top Ax \leqq \lfloor y^\top b\rfloor \text{ for each } y\in\mathbb{Q}^m_+ \text{ with } y^\top A\in\mathbb{Z}^n; \\ \\ Bx \leqq b^j; \\ (y^\top A+z^\top B)x \leqq \lfloor y^\top b+z^\top b^j\rfloor \text{ for each } y\in\mathbb{Q}^m_+,\ z\in\mathbb{Q}^m_+\setminus\{0\} \text{ with} \\ y^\top A+z^\top B\in\mathbb{Z}^n; \end{cases}$$

Note that the right hand sides of the inequalities the last two lines of (*) tend to infinity for j to infinity. Since for t=0 the lemma is obvious, the lemma follows by induction. □

PROOF OF THEOREM 2.3.3.

I. We show that the Edmonds-Johnson property is maintained under the transformations (2.3.4), and that $M(K_4)$ does not have the Edmonds-Johnson property.

Suppose A' is a bidirected graph with the Edmonds-Johnson property. This means that for each $a^1,a^2\in\mathbb{Z}^{V(A)}$; $b^1,b^2\in\mathbb{Z}^{E(A)}$ each $z\in\mathbb{R}^{V(A)}$ that is not in the integer hull of

$$(2.3.19)\quad a^1 \leqq x \leqq a^2,\ b^1 \leqq Ax \leqq b^2$$

is cut of from (2.3.19) by a Gomory cut

$$(2.3.20)\quad c^{T}x \leqq \lfloor\gamma\rfloor \quad \text{with} \quad c\in\mathbb{Z}^n \quad \text{and} \quad c^{T}x \leqq \gamma \quad \text{valid for (2.3.19).}$$

(z is *cut of by* $c^{T}x \leqq \alpha$ if $c^{T}z > \alpha$).

We now check the operations (2.3.4):

(i) Permuting rows or columns, or multiplying then by -1: trivially maintains the Edmonds-Johnson property.

(ii) Deleting a column, say corresponding to variable x_u ($u\in V(A)$): maintains the Edmonds-Johnson property (take $a_u^1=a_u^2=0$).

(iii) Deleting a row, say corresponding to edge $e\in E(A)$: maintains the Edmonds-Johnson property (take $b_e^1=-\infty$, $b_e^2=+\infty$).

(iv) Replacing $\begin{bmatrix}1 & g^{T}\\ f & D\end{bmatrix}$ by $[D-fg^{T}]$: Suppose the first matrix has the Edmonds-Johnson property. Let $\tilde{a}^1$, $\tilde{a}^2$, $\tilde{b}^1$, $\tilde{b}^2$ be integral vectors of appropriate order, and consider the systems.

$$(2.3.21)\quad \tilde{a}^1 \leqq x \leqq \tilde{a}^2,\ \tilde{b}^1 \leqq [D-fg^{T}]x \leqq \tilde{b}^2$$

and

$$(2.3.22)\quad \begin{bmatrix}-\infty\\ \tilde{a}^1\end{bmatrix} \leqq \begin{bmatrix}\lambda\\ x\end{bmatrix} \leqq \begin{bmatrix}+\infty\\ \tilde{a}^2\end{bmatrix},\ \begin{bmatrix}0\\ \tilde{b}^1\end{bmatrix} \leqq \begin{bmatrix}1 & g^{T}\\ f & D\end{bmatrix}\begin{bmatrix}\lambda\\ x\end{bmatrix} \leqq \begin{bmatrix}0\\ \tilde{b}^2\end{bmatrix}.$$

Let z be not in the integer hull of (2.3.21). It suffices to show that there exists a Gomory cut (2.3.20) violating z. To this end, define $\mu=-g^{T}z$. It is easily checked that $[\mu,z^{T}]^{T}$ is not in the integer hull of (2.3.22). Hence, by assumption there exists an inequality $\alpha\lambda+c^{T}x \leqq \delta$ valid for (2.3.22), such that $\alpha\mu+c^{T}z>\lfloor\delta\rfloor$ and α,c integral. Then $(c^{T}-\alpha g^{T})x \leqq \delta$ is valid for (2.3.21), as if x satisfies (2.3.21) then $[-g^{T}x,x^{T}]^{T}$ satisfies (2.3.22), and hence

$$(c^{T}-\alpha g^{T})x = [\alpha,c^{T}]\begin{bmatrix}-g^{T}x\\ x\end{bmatrix} \leqq \delta.$$

Similarly, $(c^{T}-\alpha g^{T})z = [\alpha,c^{T}]\begin{bmatrix}\mu\\ z\end{bmatrix} > \lfloor\delta\rfloor$, so z is cut off from (2.3.22) by a Gomory cut.

(v) $M(K_4)$ has not the Edmonds-Johnson property: Consider the system

(2.3.23) $0 \leqq x \leqq 1, \qquad 0 \leqq M(K_4)x \leqq 1.$

The integral solutions are $[0,0,0,0]^{T}$, $[1,0,0,0]^{T}$, $[0,1,0,0]^{T}$, $[0,0,1,0]^{T}$, $[0,0,0,1]^{T}$. Hence $x_1+x_2+x_3+x_4 \leqq 1$ is a facet of the integer hull of (2.3.23). However this inequality is not a Gomory cut, as $\delta=2$ is the smallest δ for which $x_1+x_2+x_3+x_4 \leqq \delta$ is valid for (2.3.23) (since $[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]^{T}$ belongs to (2.3.23)).

II. The remainder of this section is devoted to showing sufficiency in Theorem 2.3.3. Suppose the condition is not sufficient. Then there exist a bidirected graph A without an odd-K_4, and an integral vector b, such that

(2.3.24) $Ax \leqq b$

together with the odd circuit inequalities

(2.3.25) $\frac{1}{2}\sum_{e\in E(C)} x_e \leqq \left\lfloor \frac{1}{2}\sum_{e\in E(C)} b_e \right\rfloor$ (C odd circuit in A)

is not enough for determining the integer hull of (2.3.24) (since joining A with unit basis row vectors, or with the opposite of any row of A, cannot make an odd-K_4 as a subgraph). Let A be the smallest such matrix (i.e, with number of rows and columns as small as possible), and let P be the polyhedron defined by (2.3.24) and (2.3.25). Clearly A is connected, as otherwise we can decompose A and get a smaller counterexample. We may assume that in each row the sum of the absolute values of the entries is exactly 2: all-zero-rows trivially do not occur, while a row with one ± 1 can be replaced by the same row multiplied by 2.

<u>Claim 1:</u> *If* $z \in P$ *and* z *has an integral component then* z *is in the integer hull of (2.3.24).*

<u>Proof of Claim 1</u>: Suppose z_1(say) is an integer.

Let $z = \begin{bmatrix} z_1 \\ z' \end{bmatrix}$ and $A=[a_1|B]$, where a_1 is the first column of A. Then z' satisfies

$$(2.3.26) \quad Bz' \leqq b-a_1z_1.$$

We show that z' cannot be cut off from (2.3.26) by an odd circuit inequality derived from (2.3.26). For suppose $(y^{\mathsf{T}}B)x' \leqq \lfloor y^{\mathsf{T}}(b-a_1z_1)\rfloor$ is such an inequality, cutting off z', where y is 0, $\frac{1}{2}$-valued, with its $\frac{1}{2}$'s in positions corresponding to an odd circuit in B. This implies $y^{\mathsf{T}}a_1=0$. Then

$$(2.3.27) \quad (y^{\mathsf{T}}A)z = y^{\mathsf{T}}a_1z_1 + y^{\mathsf{T}}Az' = y^{\mathsf{T}}Bz' > \lfloor y^{\mathsf{T}}(b-a_1z_1)\rfloor = \lfloor y^{\mathsf{T}}b\rfloor.$$

But this is an odd circuit inequality for (2.3.24) cutting off z, contradicting the fact that z is in P.

So z' cannot be cut off from (2.3.26) by an odd circuit inequality. Hence, as B is smaller than A, z' is in the integer hull of (2.3.26), i.e. z' is a convex combination of integral solutions of (2.3.26), say of $z'_1,\ldots,z'_k$. Then z is a convex combination of the integral solutions

$$\begin{bmatrix} z_1 \\ z'_1 \end{bmatrix}, \ldots, \begin{bmatrix} z_1 \\ z'_k \end{bmatrix}$$

of (2.3.24). This proves our claim. <u>end of proof of claim 1</u>

<u>Claim 2</u>: P *has a vertex* z *with all components non-integral.*
<u>Proof of Claim 2</u>: It suffies to show that there exists a minimal face F of P such that all components of all vectors in F are non-integral (since this implies that F has dimension 0, i.e., is a vertex). In order to show this observe that P has a minimal face containing no integral vectors. If F would contain a vector z with at least one component integral, then, by Claim 1, this vector z is a convex combination of integral vectors in P, and hence in F. Contradiction.

<u>end of proof of claim 2</u>

From now, fix a vertex z with all components non-integral.

<u>Claim 3</u>: $Az < b$, *i.e.,* z *satisfies each inequality in* $Ax \leqq b$ *strictly.*
<u>Proof of Claim 3</u>: Suppose, to the contrary that the first inquality $a_1^T x \leqq b_1$ (say) satisfies $a_1^T z = b_1$ (where a_1 is the first row of A). Then a_1 contains two ± 1's: if it would contain a ± 2, and b_1 is even, Claim 2 is contradicted, while if b_1 is odd z is cut off by the odd circuit inequality obtained from a_1.

Without loss of generality we may assume $a_{11}=\pm 1$. Moreover we may assume $a_{11}=1$ (if not, multiply the first component of z and the first column of A by -1). Let

$$(2.3.28)\quad z = \begin{bmatrix} z_1 \\ z' \end{bmatrix},\ A = \begin{bmatrix} 1 & g^T \\ f & D \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b' \end{bmatrix}.$$

Then z' satisfies

$$(2.3.29)\quad [D-fg^T]x' \leqq b'-fb_1.$$

Moreover z' cannot be cut of from (2.3.29) by an odd circuit inequality derived from (2.3.29). For suppose $y^T[D-fg^T]x' \leqq \lfloor y^T(b'-fb_1)\rfloor$ is such an inequality cutting of z' from (2.3.29), with $y \geqq 0$. Then

$$(\lceil y^T f\rceil - y^T f,\ y^T)\begin{bmatrix}1 & g^T\\ f & D\end{bmatrix}\begin{bmatrix}z_1\\ z'\end{bmatrix} =$$

$$= (\lceil y^T f\rceil,\ \lceil y^T f\rceil g^T + y^T[D-fg^T])\begin{bmatrix}z_1\\ z'\end{bmatrix} =$$

$$= \lceil y^T f\rceil z_1 + \lceil y^T f\rceil g^T z' + y^T[D-fg^T]\ z' =$$

$$= \lceil y^T f\rceil b_1 + y^T[D-fg^T]z' \leqq$$

$$\leqq \lceil y^T f\rceil b_1 + \lfloor y^T(b'-fb_1)\rfloor =$$

$$= \left\lfloor(\lceil y^T f\rceil - y^T f,\ y^T)\begin{bmatrix}b_1\\ b'\end{bmatrix}\right\rfloor$$

(using $z_1+g^T z'=a_1^T z=b_1$). So z would be cut off from $Ax \leqq b$ by a Gomory cut, contradicting the fact that $z \in P$.

So z' cannot be cut off from (2.3.29) by a Gomory cut. Hence as $D-fg^T$ is smaller then A, z' is a convex combination of integral solutions of (2.3.29), say $z_1',\dots,z_k'$. Then z is a convex combination of the integral vectors

$$\begin{bmatrix}b_1 - g^T z_1'\\ z_1'\end{bmatrix},\dots,\begin{bmatrix}b_1 - g^T z_1'\\ z_k'\end{bmatrix}.$$

Each of these vectors satisfies $Ax \leqq b$, contradicting our assumption.

<u>end of proof of claim 3</u>

We call an odd circuit C *tight* if the corresponding odd circuit inequality is satisfied by z with equality, i.e., if

$$\tfrac{1}{2}\sum_{e\in E(C)} z(e) = \lfloor \tfrac{1}{2}\sum_{e\in E(C)} b_e\rfloor.$$

As z is a vertex, Claim 3 implies that z is uniquely determined by the system of equations:

$$(2.3.30)\quad \tfrac{1}{2}\sum_{e\in E(C)} x(e) = \lfloor \tfrac{1}{2}\sum_{e\in E(C)} b_e \rfloor \qquad (C \text{ tight odd circuit}).$$

Claim 4: *Each edge of* A *is in at least one tight odd circuit.*
Proof of Claim 4: If not, deleting the edge gives a smaller counterexample.

end of proof of claim 4

Without loss of generality we may assume

$$(2.3.31)\quad 0 < z_u < 1 \qquad (u\in V(A)).$$

This is allowed by replacing z by $z-\lfloor z\rfloor$ and b by $b-A\lfloor z\rfloor$. With this assumption (2.3.31) we can prove

Claim 5: $b_e = +1$ *if* e *is a* ++ *edge;*
$b_e = 0$ *if* e *is a* +- *edge;*
$b_e = -1$ *if* e *is a* -- *edge.*

Proof of Claim 5: We only show the first line - the other are similar. Let e' be a ++ edge. By Claim 3, $b_{e'} > z(e') > 0$. So $b_{e'} \geqq 1$. To show the reverse inequality, let C be a tight odd circuit containing e' (C exists by Claim 4). Let e' connect nodes u and v, say. Consider the system of linear inequalities

$$(2.3.32)\quad \begin{aligned} &x(e) \leqq b(e) \qquad (e\in E(C),\ e\neq e')\\ &x_u \leqq 1,\ x_v \leqq 1. \end{aligned}$$

For each x satisfying (2.3.32) we have

$$\tfrac{1}{2}\sum_{e\in E(C)} x(e) = \tfrac{1}{2}\sum_{e\in E(C)\setminus\{e'\}} x(e) + \tfrac{1}{2}x_u+\tfrac{1}{2}x_v \leqq 1 + \tfrac{1}{2}\sum_{e\in E(C)\setminus\{e'\}} b_e.$$

Now the constraint matrix of (2.3.32) is totally unimodular. Hence each x satisfying (2.3.32), satisfies

$$\tfrac{1}{2}\sum_{e\in E(C)} x_e \leqq 1 + \lfloor \tfrac{1}{2}\sum_{e\in E(C')\setminus\{e'\}} b_e \rfloor.$$

Since z satisfies all inequalities in (2.3.32) strictly (Claim 3 and (2.3.31)), we have

$$\lfloor \tfrac{1}{2} \sum_{e\in E(C)} b_e \rfloor = \tfrac{1}{2} \sum_{e\in E(C)} z(e) < 1 + \lfloor \tfrac{1}{2} \sum_{e\in E(C)\setminus\{e'\}} b_e \rfloor .$$

Therefore $b_{e'}<2$, and hence $b_{e'}=1$. <u>end of proof of claim 5</u>

Now recall the notions of separating and non-separating circuits given in Section 1.3 below Lemma 1.3.4.

<u>Claim 6</u>: *There are no separating tight odd circuits.*

<u>Proof of Claim 6</u>: Suppose C is such a circuit. Then we can split the edges not in C into two nonempty classes E' and E" such that if $e\in E'$ and $f\in E''$ intersect, then their common node(s) are contained in C. Let V' (V") be the set of nodes which are not in C and are covered by at least one edge in E' (E"). Consider the submatrix A' (A") of A induced by the rows $E(C)\cup E'$ and colums $V(C)\cup V'$ ($E(C)\cup E''$ and $V(C)\cup V''$). Let z' (z") be the restriction of z to $V(C)\cup V'$ ($V(C)\cup V''$). Let b' (b") be the restriction of b to $E(C)\cup E'$ ($E(C)\cup E''$).

Clearly, $A'z' \le b'$ and $A''z'' \le b''$, and z' satisfies the odd circuit inequalities for $A'x' \le b'$, and z" satisfies those for $A''x'' \le b''$. Moreover, $\tfrac{1}{2} \sum_{e\in E(C)} z'(e) = \lfloor \tfrac{1}{2} \sum_{e\in E(C)} b'_e \rfloor$, and $\tfrac{1}{2} \sum_{e\in E(C)} z''(e) = \lfloor \tfrac{1}{2} \sum_{e\in E(C)} b'_e \rfloor$, as z' and z" coincide with z on V(C) and b' and b" coincide with b on C, and as $\tfrac{1}{2} \sum_{e\in E(C)} z(e) = \lfloor \tfrac{1}{2} \sum_{e\in E(C)} b_e \rfloor$.

Since A' is smaller than A, we know that A' has the Edmonds-Johnson property. Hence z' is a convex combination of integral solutions of $A'x' \leqq b'$. Similarly, z'' is a convex combination of integral solutions of $A''x'' \le b''$. Therefore, there exists a natural number N such that

$$Nz' = z'_1+\ldots+z'_N,\ Nz'' = z''_1+\ldots+z''_N,$$

for certain integral solutions $z'_1,\ldots,z'_N$ of $A'x' \le b'$, and certain integral solutions $z''_1,\ldots,z''_N$ of $A''x'' \le b''$. Moreover we know, since

$\frac{1}{2} \sum_{e \in E(C)} x'(e) \leq \lfloor \frac{1}{2} \sum_{e \in E(C)} b_e \rfloor$, is attained by z' with equality, the same holds for $z'_1,\ldots,z'_N$. Similarly for $z''_1,\ldots,z''_N$.

Let $e_1,\ldots,e_k$ be the edges in C, and consider the corresponding inequalities (say)

$$(2.3.33) \quad x'(e_1) \leq b'_1,\ldots,x'(e_k) \leq b'_k.$$

As $\frac{1}{2} \sum_{e \in E(C)} z'_i(e) = \lfloor \frac{1}{2} \sum_{e \in E(C)} b_e \rfloor$, and $\sum_{e \in E(C)} b_e$ is odd by Claim 5, we know:

$$z'_i(e_1)+\ldots+z'_i(e_k) = b_1+\ldots b_k-1,$$

for $i=1,\ldots,N$. Hence each z'_i has equality in all constraints (2.3.33) except for one, where there is a rest of 1. Let λ'_j be the number of indices i for which z'_i has rest 1 in the j-th inequality in (2.3.33). Similarly, λ''_j is defined. Then trivially

$$z(e_1) = b_1 - \frac{\lambda'_1}{N},\ldots,\ z(e_k) = b_k - \frac{\lambda'_k}{N}.$$

Similarly, for the λ''_j. Hence $\lambda'_j = \lambda''_j$ for each j. So we may assume that z'_i and z''_i have rest 1 at the same edge in (2.3.33). As $e_1,\ldots,e_k$ are linearly independent rows of A, it follows that z'_i and z''_i are the same on V(C). So we can combine z'_i and z''_i to one integral solution z_i of $Ax \leq b$, so that z_i restricted to A' is z'_i, and z_i restricted to A" is z''_i. But then $Nz = z_1+\ldots+z_N$, contradicting our assumption that z is a non-integral vertex of P.

<u>end of proof of claim 6</u>

<u>Claim 7</u>: *Each tight odd circuit has at least three nodes of degree at least three.*

<u>Proof of Claim 7</u>: Suppose C is a tight odd circuit, with less than 3 nodes of degree at least 3. Assume C has more than 2 edges. Then C contains a node u of degree 2. If C is the only tight odd circuit containing u, we could delete u together with the two edges containing u. In the remaining bidirected graph, the remaining z_v ($v \in V(A) \setminus \{u\}$) are uniquely determined by the remaining tight odd circuits (as only one tight odd circuit is deleted). Hence we obtain a smaller counterexample.

So there exists another tight odd circuit C' containing u. As C' is non-separating, C and C' together form the whole bidirected graph. But then A has at least 3 vertices, and exactly two odd circuits, contradicting the fact that z is uniquely determined by the tight odd circuit inequalities.

Hence C has at most two edges. But then the odd circuit inequality is equivalent to $x_v \le \lfloor \frac{1}{2} \sum_{e \in E(C)} b_e \rfloor$ for a node v on C, which is tight for z, contradicting Claim 2. <u>end of proof of claim 7</u>

<u>Claim 8</u>: A *has a node* u *which is contained in each odd circuit.*

<u>Proof of Claim 8</u>: By Lemma 1.3.4 (and Claims 6 and 7) if suffices to show that if C is a tight odd circuit, then $V(C') \cap V(C) \neq \emptyset$ for each odd circuit C' in A. So it suffices to show that each two odd circuits have a node in common. Assume C' and C" are odd circuits with $V(C') \cap V(C'') = \emptyset$. As A is connected, and as each edge is contained in a tight odd circuit, there exists tight odd circuits $C_1, \ldots, C_k$ such that

$$V(C') \cap V(C_1) \neq \emptyset,\ V(C_1) \cap V(C_2) \neq \emptyset,\ V(C_2) \cap V(C_3) \neq \emptyset, \ldots,$$

$$V(C_{k-1}) \cap V(C_k) \neq \emptyset,\ V(C_k) \cap V(C'') \neq \emptyset.$$

We may assume that k is as small as possible. Hence $V(C') \cap V(C_2) = \emptyset$. So without loss of generality, $C'' = C_2$.

As C_1 is nonseparating, $V(A) \setminus V(C)$ spans a connected graph. Let T be a tree spanning $V(A) \setminus V(C)$ such that T contains all edges of E(C') and E(C") which do not intersect V(C). This is possible, as $V(C') \cap V(C'') = \emptyset$. Next delete all edges which are contained in $V(A) \setminus V(C)$ and which do not occur in T. Let A' be the bidirected graph left. Since T is bipartite, we can apply Lemma 1.3.4 to A'. It follows that V(C') and V(C") intersect, contradicting our assumption. <u>end of proof of claim 8</u>

If P is a path in A, let $P_v := (\chi^T_{E(P)} A)_v$ for each $v \in P$. A vw-path P is called *bidirected* if $P_{v'} = 0$ for each $v' \notin \{v,w\}$. So if P is a bidirected vw-path then $\sum_{e \in E(P)} x(e) = P_v x_v + P_u x_u$ and by Claim 5, $\sum_{e \in E(P)} b_e = \frac{1}{2}(P_v + P_u)$.

Claim 9: *Let* C *be a tight odd circuit, and let* v, w$\in$V(C). *If there exists a bidirected* vw-*path* P *in* A, *then there exists a bidirected* vw-*path in* C *with the same number (modulo 2) of odd edges as* P *has.*

Proof of Claim 9: Let Q_1 be the vw-path in C having the same number (modulo 2) of odd edges as P has. Let Q_2 be the other vw-path on C. Suppose Q_1 is not bidirected. Then z satisfies the following inequalities:

$$(2.3.34) \quad \tfrac{1}{2}\sum_{e\in E(Q_2)} x(e)+\tfrac{1}{2}P_v x_v+\tfrac{1}{2}P_w x_w \leqq \left\lfloor \tfrac{1}{2}\sum_{e\in E(Q_2)} b_e+\tfrac{1}{4}P_v+\tfrac{1}{4}P_w \right\rfloor$$

$$= \tfrac{1}{2}\sum_{e\in E(Q_2)} b_e+\tfrac{1}{4}P_v+\tfrac{1}{4}P_w-\tfrac{1}{2}$$

(as $E(Q_2)\Delta E(P)$ is a cycle containing an odd number of odd edges);

$$(2.3.35) \quad \tfrac{1}{2}\sum_{e\in E(Q_1)} x(e)-\tfrac{1}{2}P_v x_v-\tfrac{1}{2}P_w x_w < \left\lfloor \tfrac{1}{2}\sum_{e\in E(Q_1)} b_e-\tfrac{1}{4}P_v-\tfrac{1}{4}P_u+\tfrac{1}{2} \right\rfloor$$

$$= \tfrac{1}{2}\sum_{e\in E(Q_1)} b_e-\tfrac{1}{4}P_u-\tfrac{1}{4}P_v.$$

[Proof of (2.3.35): Consider the system:

$$x(e) \leqq b_e \quad (e\in E(Q_1)),$$

$$-P_v x_v \leqq \tfrac{1}{2} - \tfrac{1}{2}P_v,$$

$$-P_w x_w \leqq \tfrac{1}{2} - \tfrac{1}{2}P_w.$$

The constraint matrix of this system is totally unimodular. Moreover z satisfies each of the inequalities of this system strictly. Finally the left-hand side of (2.3.35) is not vanishing since Q_1 is not bidirected.] So z satisfies the sum of (2.3.34) and (2.3.35), i.e. $\tfrac{1}{2}\sum_{e\in E(C)} z_e < \tfrac{1}{2}\sum_{e\in E(c)} b_e-\tfrac{1}{2}$. This contradicts the assumption that C is tight.

end of proof of claim 9

Claim 10:

Let v$\in$V(A)\{u}. *If* P *and* Q *both are bidirected* uv-*paths, then* $P_u Q_v = P_v Q_u$.

Proof of Claim 10: Suppose $P_u Q_v \neq P_v Q_u$. Then $E(P)\Delta E(Q)$ is a cycle containing an odd number of odd edges. Hence it follows that

$$P_u z_u + P_v z_v + Q_u z_u + Q_v z_u = \sum_{e\in E(P)} z(e) + \sum_{e\in E(Q)} z_e =$$

$$= \sum_{e\in E(P)\Delta E(Q)} z(e) + 2 \sum_{e\in E(P)\cap E(Q)} z_e \leqq$$

$$= \sum_{e\in E(P)\Delta E(Q)} b_e - 1 + 2 \sum_{e\in E(P)\cap E(Q)} b_e =$$

$$= \sum_{e\in E(P)} b_e + \sum_{e\in E(Q)} b_e - 1 = \tfrac{1}{2}P_u + \tfrac{1}{2}P_v + \tfrac{1}{2}Q_u + \tfrac{1}{2}Q_v - 1.$$

Without no loss of generality we may assume $P_u = P_v$, $Q_u = -Q_v$. So $P_u(z_u+z_v) + Q_u(z_u-z_v) \leqq P_u - 1$. Now one easily verifies that each possible choice of $P_u \in \{1,-1\}$ and $Q_u \in \{1,-1\}$ contradicts (2.3.31), saying that $0 < z_v < 1$ and $0 < z_u < 1$. <u>end of proof of claim 10</u>

By Claim 10 the following vector $\tilde{z} \in \mathbb{R}^{V(A)}$ is well-defined

$$\tilde{z}_v = \begin{cases} 2 & v=u; \\ -\dfrac{P_u}{P_v} & v\neq u,\ \ P \text{ is a bidirected uv-path;} \\ 0 & \text{else.} \end{cases}$$

<u>Claim 11</u>: *If* C *is a tight odd circuit, then* $\sum_{e\in E(C)} \tilde{z}(e) = 0$

<u>Proof of Claim 11</u>: By Claims 9 and 10, C exists of three edge disjoint paths P, Q, and R such that $V(P)\cap V(Q)=\{u\}$, P and Q are bidirected, $v\neq w$ (where $\{v\} := V(P)\cap V(R)$, $\{w\} := V(Q)\cap V(R)$) and $\tilde{z}_{v'}=0$ for all $v' \in V(R)\setminus\{v,w\}$. From this of follows that

$$\sum_{e\in E(C)} \tilde{z}(e) = \sum_{e\in E(P)} \tilde{z}(e) + \sum_{e\in E(R)} \tilde{z}(e) + \sum_{e\in E(Q)} \tilde{z}(e) =$$

$$= P_u\tilde{z}_u + P_v\tilde{z}_v + R_v\tilde{z}_v + R_w\tilde{z}_w + Q_w\tilde{z}_w + Q_u\tilde{z}_u =$$

$$= 2P_u - P_u - P_u - Q_u - Q_u + 2Q_u = 0.$$

(Here we used that $R_v = P_v$, $R_w = Q_w$.) <u>end of proof of claim 11</u>

As $\tilde{z} \neq 0$, Claim 11 contradicts the fact that z is a vertex of P determined by the tight odd circuit inequalities uniquely. This contradiction finishes the proof of Theorem 2.3.3. $\square$

CHAPTER 3. SIGNED GRAPHS WITH NO ODD-K_4

Motivated by the main result of Section 2.3 (Theorem 2.3.3), in this section we study combinatorial properties of signed graphs with no odd-K_4, and related types of signed graphs. First we show, in Section 3.2, that each signed graph with no odd-K_4 can be obtained by glueing together certain "elementary" signed graphs. Next, in Section 3.4, we prove that a signed graph has no odd-K_4 and no so-called odd-K_3^2 (cf. Section 3.1) if and only if some specific orientation of the edges exists. Using this we give in Sections 3.5 and 3.6 new proofs of results due to Seymour, Gerards and Catlin. In Section 3.7, we use the results of Sections 3.2 and 3.4 to prove a new result extending König's well-known theorems on stable sets and node-covers in bipartite graphs to graphs with no odd-K_4.

In proving the results in Section 3.2 and 3.3 we use the theory of regular matroids. The relation between regular matroids and signed graphs with no odd-K_4 and no odd-K_3^2 is elaborated in Section 3.1.

3.1. SIGNED GRAPHS AND BINARY MATROIDS

Let (G,Σ) be a signed graph. The matroid $\mathcal{S}(G,\Sigma)$ is the binary matroid represented over GF(2) by:

$$(3.1.1) \quad \left[\begin{array}{c|c} 1 & \chi_\Sigma^T \\ \hline 0 & \\ \vdots & M_G \\ 0 & \end{array}\right]$$

where M_G is the node-edge incidence matrix of G, and χ_Σ the characteristic vector of Σ, as a subset of $E(G)$. Throughout this chapter we denote the element of $E(\mathcal{S}(G,\Sigma))$ corresponding to the first column of (3.1.1) by p. So $E(\mathcal{S}(G,\Sigma)) = \{p\} \cup E(G)$. The motivation for definining $\mathcal{S}(G,\Sigma)$ is the following theorem, which is the main observation of this section.

<u>Theorem 3.1.2</u>

Let (G,Σ) *be a signed graph.*

(i) The following are equivalent:

- (G,Σ) *contains no odd-*K_4*;*
- $\mathcal{S}(G,\Sigma)$ *has no* F_7^**-minor using* p.

(ii) The following are equivalent:

- (G,Σ) *contains no odd-*K_4 *and no odd-*K_3^2*;*
- $\mathcal{S}(G,\Sigma)$ *is regular.* □

The term *odd-*K_3^2 used in this theorem (which we prove later in this section) stands for signed graphs of the form depicted in Figure 3.1. Here wriggled and dotted lines stand for pairwise openly disjoint paths. Wriggled lines must have at least one edge. (Dotted lines may have length zero.) The symbol **odd** in Figure 3.1 indicates that the bounding circuit of the corresponding face is an odd circuit.

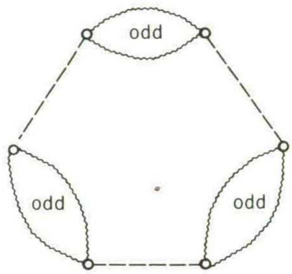


Figure 3.1: odd-K_3^2

Before we prove Theorem 3.1.2 and related assertions, we discuss the circuits, rank function, and minors of $\mathcal{S}(G,\Sigma)$.

CIRCUITS OF $\mathcal{S}(G,\Sigma)$

The circuits of $\mathcal{S}(G,\Sigma)$ are the sets of the following forms:

- $E(C)$, if C is an even circuit in (G,Σ),
- $E(C)\cup\{p\}$, if C is an odd circuit in (G,Σ),
- $E(C_1)\cup E(C_2)$, if both C_1 and C_2 are odd circuits in (G,Σ) such that $|V(C_1)\cap V(C_2)|\leqq 1$.

RANKFUNCTION OF $\mathcal{S}(G,\Sigma)$

Let $E'\subset E(G)$. Then the following hold:

(3.1.3) $r_{\mathcal{S}(G,\Sigma)}(E'\cup\{p\}) = r_{\mathcal{M}(G)}(E') + 1$;

(3.1.4) $$r_{\mathcal{S}(G,\Sigma)}(E') = \begin{cases} r_{\mathcal{M}(G)}(E')+1 & \text{if } E' \text{ contains an odd circuit in } (G,\Sigma);\\ r_{\mathcal{M}(G)}(E') & \text{if } E' \text{ does not contain an odd circuit in } (G,\Sigma).\end{cases}$$

MINORS OF $\mathcal{S}(G,\Sigma)$ VERSUS REDUCTIONS OF (G,Σ)

There is a strong connection between reductions of (G,Σ) and minors of $\mathcal{S}(G,\Sigma)$. First, it should be noted that resigning (G,Σ) does not change $\mathcal{S}(G,\Sigma)$; i.e. $\mathcal{S}(G,\Sigma\Delta\delta(U)) = \mathcal{S}(G,\Sigma)$ for any $U\subset V(G)$. Moreover we have:

- $\mathcal{S}(G,\Sigma)\backslash e = \mathcal{S}(G\backslash e,\Sigma\backslash\{e\})$ if $e\in E(G)$;
- $\mathcal{S}(G,\Sigma)/e = \mathcal{S}(G/e,\Sigma)$ if $e\in E(G)\backslash\Sigma$, and e is not a loop in G;

- $\mathcal{S}(G,\Sigma)/e = \mathcal{S}(G/e,\Sigma\Delta\delta(u))$ if $e\in\Sigma$, e is not a loop in G, and u is an endpoint of e.

In case e is a loop:
$\mathcal{S}(G,\Sigma)/e = \mathcal{S}(G\backslash e,\Sigma)$ if $e\in E(G)\backslash\Sigma$;
$\mathcal{S}(G,\Sigma)/e \sim \mathcal{S}(G,\Sigma)/p$ if $e\in\Sigma$ (since then e is parallel with p in $\mathcal{S}(G,\Sigma)$).

To be complete:
$\mathcal{S}(G,\Sigma)\backslash p$ is the binary matroid with cycle space $\{E(C) \mid C$ is a cycle in G and $|E(C)\cap\Sigma|$ is even$\}$;
$\mathcal{S}(G,\Sigma)/p = \mathcal{M}(G)$.

Since the only "minor-minimal" non-regular matroids are F_7 and F_7^* (Tutte [1958], cf. Theorem 1.4.4), we want to know how F_7 and F_7^* arise as binary matroids of type $\mathcal{S}(G,\Sigma)$.

<u>Lemma 3.1.5</u>
Let (G,Σ) *be a signed graph, Then:*
(i) $\mathcal{S}(G,\Sigma) \sim F_7$ *if and only if* $(G,\Sigma) \sim \tilde{K}_3^2$*;*
(ii) $\mathcal{S}(G,\Sigma) \sim F_7^*$ *if and only if* $(G,\Sigma) \sim \tilde{K}_4$. □

Here $\tilde{K}_3^2$ denotes the signed graph in Figure 3.2 (bold edges are odd, thin edges are even). $\tilde{K}_4 = (K_4,E(K_4))$ (cf. Section 1.3).

Figure 3.2

The proof of Lemma 3.1.5 is easy, as is the proof of the following extension.

Lemma 3.1.6

Let (G,Σ) *be a signed graph. Then the following hold:*

(i) $\mathcal{S}(G,\Sigma)\backslash p \sim F_7$ *if and only if* (G,Σ) *is equivalent with one of the two signed graphs in Figure 3.3(a);*

(ii) $\mathcal{S}(G,\Sigma)\backslash p \sim F_7^*$ *if and only if* (G,Σ) *is equivalent to one of the three signed graphs in Figure 3.3(b). (Bold edges in Figure 3.3 are odd, and so are loops. Thin edges are even.)* □

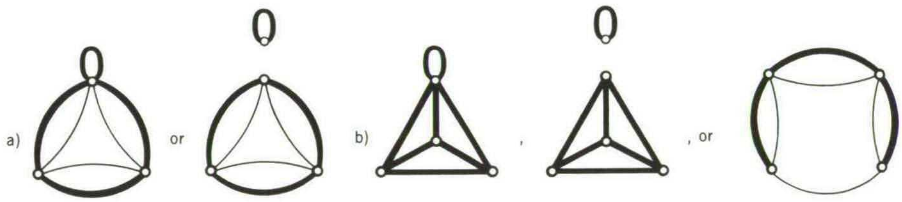


Figure 3.3

Using Lemma 3.1.5 and the relation between minors of $\mathcal{S}(G,\Sigma)$ and reductions of (G,Σ) we can easily prove the following result.

Lemma 3.1.7

Let (G,Σ) *be a signed graph.*

(i) The following are equivalent:

- $\mathcal{S}(G,\Sigma)$ *has an* F_7*-minor using* p;
- (G,Σ) *reduces to* $\tilde{K}_3^2$.

(ii) The following are equivalent:

- $\mathcal{S}(G,\Sigma)$ *has an* F_7^**-minor using* p;
- (G,Σ) *reduces to* $\tilde{K}_4$. □

We now prove Theorem 3.1.2.

<u>Proof of Theorem 3.1.2</u>

Theorem 3.1.2(i) follows directly from Lemma 3.1.5(ii) and Lemma 1.3.3.

To prove Theorem 3.2.1(ii), first observe the following (easy-to-prove) property:

(3.1.8) (G,Σ) contains no odd-K_4 and no odd-K_3^2 if and only if (G,Σ) reduces neither to $\tilde{K}_4$ nor to $\tilde{K}_3^2$.

Using this, together Lemma 3.1.5, and Lemma 3.1.6, 3.1.2(ii) easily follows. □

<u>Remark</u>:

In view of Lemma 1.3.3 and the equivalence (3.1.8) one might expect the following to be true: "(G,Σ) contains an odd-K_3^2 if and only if (G,Σ) reduces to $\tilde{K}_3^2$". However it is not, as the signed graph in Figure 3.4 shows (bold edges are odd, thin edges are even).

Figure 3.4

3.2. DECOMPOSITIONS

There are two special types of signed graphs which do not have an odd-K_4 or an odd-K_3^2. These types are:

ALMOST BIPARTITE SIGNED GRAPHS

A signed graph (G,Σ) is called *almost bipartite* if there exists a node $u \in V(G)$ such that $u \in V(C)$ for each odd circuit C.

PLANAR SIGNED GRAPHS WITH TWO ODD FACES

A signed graph is *planar with two odd faces* if it can be embedded in the plane such that all but two faces have a bounding circuit that is even.

The fact that all graphs of either type have no odd-K_4 and no odd-K_3^2 is easy to see. In fact, in a sense these are the only examples of signed graphs with no odd-K_4 and no odd-K_3^2. If such signed graph is not one of the above types, it can be decomposed into smaller signed graphs with no odd-K_4 and no odd-K_3^2 (Theorem 3.2.3). A similar result holds for signed graphs with no odd-K_4 (Theorem 3.2.4) and for signed graphs with no odd-K_3^2 (Theorem 3.2.6). Theorem 3.2.3 yields a polynomial-time algorithm to recognize whether or not a given signed graph contains an odd-K_4 or an odd-K_3^2.

To prove Theorem 3.2.3, we use the following famous result of Seymour [1980].

<u>Theorem 3.2.1</u> (Seymour [1980])
Let $\mathcal{M}$ be a regular matroid. Then at least one of the following holds:
(1) *There exists a partition* $X_1 \cup X_2$ *of* $E(\mathcal{M})$ *such that*

$$r_{\mathcal{M}}(X_1) + r_{\mathcal{M}}(X_2) \leqq r_{\mathcal{M}}(E(\mathcal{M})) + k-1,$$

where $k = 1,2$ *and* $|X_1|,\ |X_2| \geqq k$,
or $k = 3$ *and* $|X_1|,\ |X_2| \geqq 6$.

(2) $\mathcal{M}$ *is graphic, co-graphic, or isomorphic to the matroid, called* $\mathcal{R}_{10}$*, which is represented over* GF(2) *by the matrix*

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{array}\right].$$

□

Remarks:

Seymour, [1980], states his result slightly different: In (1) he only requires $|X_1|$, $|X_2| \geqq 4$ if $k = 3$. However, using the statements (7.4), (9.2) and (14.2) of his paper one can sharpen this to $|X_1|$, $|X_2| \geqq 6$ if $k = 3$. We use this in proving Theorem 3.2.3. Note that $\mathcal{R}_{10} \sim \mathcal{S}(K_5, E(K_5))\backslash p$, where K_5 denotes the complete graph on five nodes.

Important in the decomposition of signed graphs with no odd-K_4 and no odd-K_3^2 is the notion of so-called *splits*.

Assume E_1, E_2 are nonempty subsets of $E(G)$, partitioning $E(G)$. Denote the set of nodes spanned by E_1, and E_2 respectively, by V_1, V_2 respectively. $\bar{G}_i$ is defined by $V(\bar{G}_i) := V_i$, $E(\bar{G}_i) = E_i$ for $i = 1,2$.

1-SPLIT:

Let $|V_1 \cap V_2| \leq 1$. Then $(\bar{G}_1, E_1 \cap \Sigma)$ and $(\bar{G}_2, E_2 \cap \Sigma)$ are said to form a 1-*split* of (G, Σ). $(\bar{G}_1, E_1 \cap \Sigma)$ and $(\bar{G}_2, E_2 \cap \Sigma)$ are the *parts* of the 1-split.

2-SPLIT:

Let $|V_1 \cap V_2| = 2$, $V_1 \cap V_2 = \{u, v\}$, say. Moreover, let for $i = 1, 2$, $\bar{G}_i$ be connected and not a signed subgraph of the signed graph in Figure 3.5.

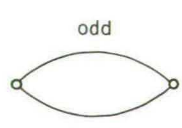


Figure 3.5

Define (G_1,Σ_1) as follows: If $(\bar{G}_2,E_2\cap\Sigma)$ is not bipartite, add to $(\bar{G}_1,E_1\cap\Sigma)$ the two edges in Figure 3.5. If $(\bar{G}_2,E_2\cap\Sigma)$ is bipartite, add a single edge e from u to v. Take $e\in\Sigma_1$ if and only if there exists an odd uv-path in G_2. (A path is *odd* if it contains an odd number of odd edges.) (G_2,Σ_2) is defined analogously. Now (G_1,Σ_1) and (G_2,Σ_2) are said to form a *2-split* of (G,Σ). The signed graphs (G_1,Σ_1) and (G_2,Σ_2) are called the *parts* of the 2-split. If $(\bar{G}_i,E_i\cap\Sigma)$ is not bipartite for i = 1, 2, then we call the 2-split *strong*. In Figure 3.6, we give an example of a strong 2-split.

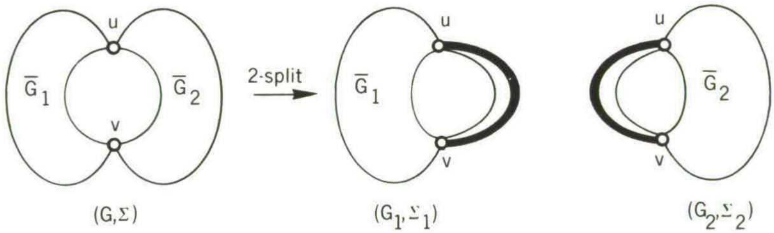


Figure 3.6

3-SPLIT:

Let $|V_1\cap V_2| = 3$, $V_1\cap V_2 = \{u_1,u_2,u_3\}$ say. Moreover, let $\bar{G}_2$ be bipartite and connected. Finally, let $|E_2| \geq 4$. Define G_1 as follows: $V(G_1) := V_1\cup\{\tilde{v}\}$ (where $\tilde{v}$ is a new node), and $E(G_1) := E_1\cup\{u_1\tilde{v},\ u_2\tilde{v},\ u_3\tilde{v}\}$. $\tilde{E}$ is the subset of $\{u_2\tilde{v},\ u_3\tilde{v}\}$ defined by: $u_i\tilde{v}\in\tilde{E}$ if and only if there exists an odd path from u_1 to u_i in $(\bar{G}_2,E_2\cap\Sigma)$ (i=2,3). We define $\Sigma_1 := (E_1\cap\Sigma)\cup\tilde{E}$. Now (G_1,Σ_1) is said to form a *3-split* of (G,Σ). (G_1,Σ_1) is called the *part* of the 3-split. So a 3-split has one part only.

<u>Lemma 3.2.2</u>

Let (G,Σ) be a signed graph with a k-split ($k \leq 3$) and no ℓ-split for any $\ell < k$. Then the following hold:

(i) (G,Σ) contains no odd-K_4 if and only if each part of the k-split contains no odd-K_4;

(ii) (G,Σ) *does not reduce to* $\tilde{K}_3^2$ *if and only if each part of the* k*-split does not reduce to* $\tilde{K}_3^2$.

Proof: Straightforward. (Note that if (G,Σ) has a k-split ($k \leqq 3$) and no ℓ-spit for any $\ell < k$, then each part of the k-split is a reduction of (G,Σ).) □

Next we arive at the main result of this section.

Theorem 3.2.3

Let (G,Σ) *be a signed graph, with no odd-*K_4 *and no odd-*K_3^2.
Then at least one of the following holds:

(i) (G,Σ) *has a 1-, 2-, or 3-split;*

(ii) (G,Σ) *is almost bipartite;*

(iii) G *is planar with at most two odd faces (with respect to* Σ*);*

(iv) (G,Σ) *is equivalent with the signed graph in Figure 3.7 below. (Thin edges are even, bold edges are odd.).*

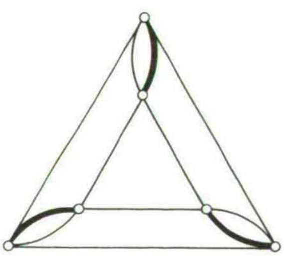


Figure 3.7

Remarks:

If (G,Σ) satisfies (ii), then $\mathcal{S}(G,\Sigma)$ is graphic. Similarly, if (G,Σ) satisfies (iii), then $\mathcal{S}(G,\Sigma)$ is co-graphic. (Note that the reverse implications do not hold in general.) If (G,Σ) satisfies (i), then $\mathcal{S}(G,\Sigma)$ satisfies (1) of Theorem 3.2.1. However (iv) has no relation with $\mathcal{R}_{10}$. (In fact $\mathcal{R}_{10} \not\prec \mathcal{S}(G,\Sigma)$ for each signed graph (G,Σ).) $\mathcal{S}(\tilde{G},\tilde{\Sigma})$ satisfies Theorem 3.2.1 (2) with $k = 3$, in case $(\tilde{G},\tilde{\Sigma})$ is the signed graph of Figure 3.7. Indeed, let E_1 be the set of edges of the outer and the inner triangle (cf. Figure 3.7), and $E_2 := E(\tilde{G})\setminus E_1$.

Then

$$r_{\mathcal{S}(\tilde{G},\tilde{\Sigma})}(E_1) + r_{\mathcal{S}(\tilde{G},\tilde{\Sigma})}(E_2\cup\{p\}) = 4 + 4 = r_{\mathcal{S}(\tilde{G},\tilde{E})}(E(\tilde{G})) + 2.$$

(This also follows, indirectly, by Seymour [1980:(9.2)] since $\mathcal{S}(\tilde{G},\tilde{\Sigma})\backslash p \sim \mathcal{R}_{12}$.) On the other hand $(\tilde{G},\tilde{\Sigma})$ has no 1-, 2-, or 3-split.

<u>Proof</u>: Let (G,Σ) be a signed graph with no odd-K_4 and no odd-K_3^2. Assume (G,Σ) has no 1-, 2-, or 3-split. Since $\mathcal{S}(G,\Sigma)$ is regular (Lemma 3.1.2 (ii)), we can apply Seymour's theorem (Theorem 3.2.1). We consider four cases:

<u>Case I</u>: $\mathcal{S}(G,\Sigma)$ *is graphic.*

Let $\tilde{G}$ be an undirected graph, such that $\mathcal{M}(\tilde{G}) \sim \mathcal{S}(G,\Sigma)$. Denote the edge in $E(\tilde{G})$ corresponding to p by e_p. Then $\mathcal{M}(G) = \mathcal{S}(G,\Sigma)/p \sim \mathcal{M}(\tilde{G})/e_p = \mathcal{M}(\tilde{G}/e_p)$. As G is 3-connected, $G \sim \tilde{G}/e_p$ (Whitney [1933]). We might as well assume that $\tilde{G}$ is such that $G = \tilde{G}/e_p$. Taking $v_0 \in V(G)$ equal to the node in which e_p is contracted, we easily see that (G,Σ) is almost bipartite.

<u>Case II</u>: $\mathcal{S}(G,\Sigma)$ *is co-graphic.*

Let $\tilde{G}$ be an undirected graph, such that $\mathcal{M}(\tilde{G})^* \sim \mathcal{S}(G,\Sigma)$. Then $\mathcal{M}(G) = \mathcal{S}(G,\Sigma)/p \sim \mathcal{M}(\tilde{G})^*/p = \mathcal{M}(\tilde{G}\backslash e_p)^*$ ($e_p \in E(\mathcal{M}(\tilde{G}))$ corresponds to p). So G is planar. As G is 3-connected we may assume, as in Case 1 above, that $\tilde{G}\backslash e_p$ is the planar dual of G. It is not hard to see that the only odd faces of G (with respect to Σ) are the faces of G corresponding to the endnodes of e_p in $\tilde{G}$, which proves that $\mathcal{S}(G,\Sigma)$ satisfies (iii).

<u>Case III</u>: $\mathcal{S}(G,\Sigma) \sim \mathcal{R}_{10}$.

For any $x \in E(\mathcal{R}_{10})$ we have $\mathcal{R}_{10}/x \sim \mathcal{M}^*(K_{3,3})$. Since $\mathcal{S}(G,\Sigma)/p$ is graphic, this implies that Case III cannot occur.

Case IV: *$\mathcal{S}(G,\Sigma)$ satisfies (1) of Theorem 3.2.1.*

Assume $\mathcal{S}(G,\Sigma)$ does not satisfy (ii) or (iii) of Theorem 3.2.3. Let E_1 and E_2 partition $E(G)$ such that

$$(*)\quad r_{\mathcal{S}(G,\Sigma)}(E_1) + r_{\mathcal{S}(G,\Sigma)}(E_2\cup\{p\}) = r_{\mathcal{S}(G,\Sigma)}(E(G)\cup\{p\}) + k-1$$

with $k = 1,2$ and $|E_1| \geqq k$, $|E_2| \geqq k-1$,
or $k = 3$ and $|E_1| \geqq 6$, $|E_2| \geqq 5$.

Let $\varepsilon := 0$ if E_1 is bipartite, and $\varepsilon := 1$ if E_1 is not bipartite. Then, by (3.1.3) and (3.1.4), (*) is equivalent to:

$$(**)\qquad r_{\mathcal{M}(G)}(E_1) + r_{\mathcal{M}(G)}(E_2) = r_{\mathcal{M}(G)}(E(G)) + (k-\varepsilon)-1.$$

If $|E_2| = 0$, then $k \leqq |E_2| + 1 \leqq 1$. Hence, by (**): $\varepsilon = 0$. So (G,Σ) is bipartite, which implies (iii). Therefore we may assume $|E_2| \geqq 1$. Consider the two subgraphs G_1 and G_2 of G with $V(G_1) = V(G_2) = V(G)$, $E(G_1) = E_1$, and $E(G_2) = E_2$. Let $E_1^1,\ldots,E_1^s$; $E_2^1,\ldots,E_2^t$ be the edge-sets of the components of G_1; G_2 respectively. Define the undirected graph H as follows:

> $V(H) := \{u_1,\ldots,u_s, v_1,\ldots,v_t\}$,
> for each $v\in V(G)$ there exists an edge from u_i to v_j if v is spanned by E_1^i and by E_2^j $(i=1,\ldots,s;\ j=1,\ldots,t)$.

(So H may have parallel edges). For $i=1,2$, let V_i be the set of nodes in $V(G_i)$ that are not isolated.

<u>Claim 1</u>: $|E(H)| = s + t + k - \varepsilon - 2 = |V(H)| + k - \varepsilon - 2.$
<u>Proof of Claim 1</u>: $r_{\mathcal{M}(G)}(E_1) = |V_1| - s$, $r_{\mathcal{M}(G)}(E_2) = |V_2| - t$ and $r_{\mathcal{M}(G)}(E) = |V(G)| - 1$. ($G$ is connected, as (G,Σ) has no 1-split.) Since $|V_1\cap V_2| = |E(H)|$ and $|V_1\cup V_2| = |V(G)|$, (**) yields the claim.

<u>end of proof of claim 1</u>

Claim 2: H *is a bipartite, connected graph, without isthmuses.* (*An isthmus is a coboundary consisting of one edge only.*)

Proof of Claim 2: By definition, H is bipartite. If H is disconnected, or has an isthmus, then (G,Σ) has a 1-split.

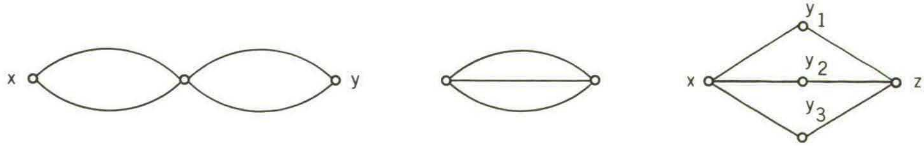
end of proof of claim 2

Claim 3: *H has no two adjacent nodes of degree* 2.

Proof of Claim 3: Assume, to the contrary, that u_i and v_j are adjacent nodes of H, both of degree 2. If between u_i and v_j there are parallel edges, then by Claim 2: $V(H) = \{u_i,v_j\}$. So $i = j = s = t = 1$. By Claim 1: $k - \varepsilon = 2$. Since (G,Σ) has no 2-split, E_1 or E_2 is contained in the signed graph of Figure 3.5. Hence $|E_1| \leqq 2$ or $|E_2| \leqq 2$. So, from (*), $k \leqq 2$. As $k - \varepsilon = 2$ it follows that $k = 2$ and $\varepsilon = 0$. Hence E_1 is bipartite and E_2 the signed graph of Figure 3.5. So (G,Σ) is almost bipartite. Contradiction.

end of proof of claim 3

Claim 4: $k = 3$, $\varepsilon = 0$, *and* H *is the graph in Figure 3.8(c) below.*



(a) (b) (c)

Figure 3.8

Proof of Claim 4: By Claims 2 and 3: $|E(H)| \geqq |V(H)| + 1$. Hence by Claim 1: $k - \varepsilon - 2 \geqq 1$. So $k = 3$, and $\varepsilon = 0$. From the previous claims, it follows that H is (isomorphic to) one of the graphs in Figure 3.8(a), (b), and (c). So it remains to show that H cannot be one of the graphs in Figure 3.8(a) and (b). Since $k = 3$ we have $|E_1| \geqq 6$ and $|E_2| \geqq 5$. If H is the graph in Figure 3.8(a), then either node x, or node y in Figure 3.8(a), corresponds to an E_1^i or E_2^i with at least three elements. This would yield a 2-split. If H is the graph in Figure 3.8(b), then we have a 3-split (E_1 is bipartite, as $\varepsilon = 0$), which is a contradiction.

<u>end of proof of claim 4</u>

We conclude by investigating the case that H equals the graph in Figure 3.8(c). If nodes y_1, y_2 and y_3 correspond to E_1^1, E_1^2 and E_1^3 respectively, then we have a 2-split. Indeed, at least one of E_1^i has cardinality at least 2 (as $|E_1| \geqq 6$), and is therefore not contained in the signed graph of Figure 3.5 (as E_1 is bipartite). So y_1, y_2 and y_3 correspond to E_2^1, E_2^2, and E_2^3 respectively. Since (G,Σ) has no 3-split, both $|E_1^1|$ and $|E_1^2|$ are at most 3. So, as $|E_1| \geqq 6$, $|E_1^1| = |E_1^2| = 3$. Moreover both E_1^1 and E_1^2 are triangles, as, otherwise, (G,Σ) has a 2-split. For the same reason E_2^1, E_2^2 and E_2^3 are contained in the signed graph of Figure 3.7. As (G,Σ) does not satisfy (iii), (G,Σ) is equal to the signed graph of Figure 3.7. So (iv) follows. $\square$

<u>Remark</u>:

The proof technique used in Case IV of the proof above is also used by Truemper [1986] to characterize those partitions E_1, E_2 of the edge-set of a k-connected graph G, that satisfy:

$$\begin{cases} r_{\mathcal{M}(G)}(E_1) + r_{\mathcal{M}(G)}(E_2) \leqq r_{\mathcal{M}(G)}(E(G)) + k-1 \\ \\ |E_1|,\ |E_2| \geqq k. \end{cases}$$

Also the class of signed graphs with no odd-K_4 can be characterized in a way similar to Theorem 3.2.3. This is stated in the following result, first stated by Lovász, Seymour, Schrijver, and Truemper [private communication].

<u>Theorem 3.2.4</u>

Let (G,Σ) be a signed graph. Then (G,Σ) contains no odd-K_4 if and only if one of the following holds:

(i) (G,Σ) *is almost bipartite, planar with two odd faces, equivalent with the signed graph of Figure 3.5, or equivalent with* $\tilde{K}_3^2$;

(ii) (G,Σ) *has a* k-*split* ($k \leqq 3$), *no* ℓ-*split for* $0 \leqq \ell < k$, *and each part of the* k-*split contains no odd*-K_4. □

This result, as well as Theorem 3.2.6., is a special instance of a theorem of Truemper and Tseng [1986]: Let $\mathcal{M}$ be a binary matroid, and $x \in E(\mathcal{M})$, if $\mathcal{M}$ does not have an F_7-minor using x then either $\mathcal{M}$ is regular, or $\mathcal{M} \sim F_7^*$, or $\mathcal{M}$ satisfies (1) of Theorem 3.2.1.

Theorem 3.2.4 easily follows from Lemma 3.2.2, Theorem 3.2.3 and the following result, observed by Lovász and Schrijver.

<u>Theorem 3.2.5</u>

Let (G,Σ) *be a signed graph with no odd*-K_4. *Then one of the following holds:*

(i) (G,Σ) *has a* 1-*split or a strong* 2-*split;*

(ii) $(G,\Sigma) \sim \tilde{K}_3^2$*;*

(iii) (G,Σ) *contains no odd*-K_3^2.

<u>Proof</u>: Let (G,Σ) be a signed graph with no odd-K_4. Suppose (G,Σ) contains no strong 2-split, but does contain an odd-K_3^2. Let $(\tilde{G},\tilde{\Sigma})$ be an odd-K_3^2 contained in (G,Σ) such that $|E(P_1)| + |E(P_2)| + |E(P_3)|$ is minimal. (P_1, P_2 and P_3 are the paths indicated in Figure 3.9.)

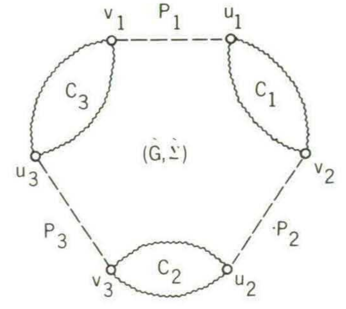


Figure 3.9

The odd circuits C_1, C_2 and C_3, as well as the nodes v_1, v_2, v_3, u_1, u_2, and u_3 are as indicated in Figure 3.9. (Note that v_i may be equal to

u_i (i=1,2,3).) Define: $V_i := V(P_i) \cup V(C_i)$ (i=1,2,3). If $S \subseteq V(G)$, then a path P from u to v is called an *S-path* if $V(P) \cap S = \{u,v\}$.

Claim: *If* P *is a* $V(\tilde{G})$*-path, then* P *is a* V_i*-path, for* i=1,2 *or* 3.
Proof of Claim: Let P be a $V(\tilde{G})$-path. Let u and v be the endpoints of P. Assume P is not a V_i-path (i=1,2,3). Hence we may assume $v \notin \{v_1, v_2, v_3\}$. Moreover we may assume $v \in V_2$. So $u \notin \{v_2, v_3\}$. Finally we may assume $u \in V_1$. (Indeed, if $u \notin V_1$, then $u \neq v_1$. Interchanging u and v, and renumbering indices yields $u \in V_1$, $v \in V_2$.) We consider three cases.

Case I: $v \in V(C_2) \setminus \{u_2\}$.

Then $\tilde{G}$ and P together contain an odd-K_4. This yields a contradiction.

Case II: $u \in V(P_1)$ *and* $v \in V(P_2)$.

Then $\tilde{G}$ and P together contain a odd-K_3^2 with smaller $|E(P_1)| + |E(P_2)| + |E(P_3)|$. Again we have a contradiction.

Case III: $u \in V(C_1) \setminus \{u_1\}$ *and* $v \in V(P_2)$.

Now there are two possibilities. If the circuit C (see Figure 3.10) is odd then $\tilde{G}$ and P together contain an odd-K_4. If C is even we find an odd-K_3^2 with smaller $|E(P_1)| + |E(P_2)| + |E(P_3)|$. So both possibilities yield a contradiction.

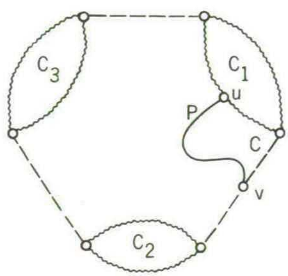


Figure 3.10

end of proof of claim

Since (G,Σ) has no strong 2-split, the claim yields, for i=1,2,3: $E(P_i) = \emptyset$, and C_i consists of two parallel edges, one odd and one even. So $(\tilde{G},\tilde{\Sigma}) \sim \tilde{K}_3^2$. If $V(G) = V(\tilde{G})$ then, as (G,Σ) has no 1-split and no strong 2-split: $(G,\Sigma) = (\tilde{G},\tilde{\Sigma}) \sim \tilde{K}_3^2$ and the theorem is proved. So let us suppose: $V(G) \neq V(\tilde{G})$. Let $v \in V(G)\setminus V(\tilde{G})$. There are three internally node disjoint paths Q_1, Q_2 and Q_3 each going from v to a different node on $\tilde{G}$ (as (G,Σ) has no 1-split and no strong 2-split). But this is impossible since then $\tilde{G}$, Q_1, Q_2 and Q_3 together contain an odd-K_4. □

Finally we state a decomposition result for signed graphs which do not reduce to $\tilde{K}_3^2$.

Theorem 3.2.6
Let (G,Σ) be a signed graph. Then (G,Σ) does not reduce to $\tilde{K}_3^2$ if and only if one of the following holds:
(i) (G,Σ) is almost bipartite, or planar with two odd faces, or equivalent with the signed graph of Figure 3.7, or equivalent with $\tilde{K}_4$;
(ii) (G,Σ) has a k-split ($k \leqq 3$), and no ℓ-split for $0 < \ell \leqq k$. Moreover each part of the k-split does not reduce to $\tilde{K}_3^2$. □

This theorem follows from Lemma 3.2.2, Theorem 3.2.3, and the following lemma.

Lemma 3.2.7
Let (G,Σ) be a signed graph without a 1-, or 2-split, such that (G,Σ) does not reduce to $\tilde{K}_3^2$. Then $(G,\Sigma) \sim \tilde{K}_4$, or (G,Σ) contains no odd-K_4.

Proof: The lemma follows from the following two results.
Let $\mathcal{M}$ be a binary matroid, and $x \in E(\mathcal{M})$.
(a) If $\mathcal{M}$ contains no F_7-minor using x, and $\mathcal{M}/x$ contains no F_7-minor, then $\mathcal{M}$ contains no F_7-minor at all.
(b) If $\mathcal{M}$ is 3-connected, and contains no F_7-minor, then either $\mathcal{M}$ is regular, or $\mathcal{M} \sim F_7$. ($\mathcal{M}$ is *3-connected* means: $\mathcal{M}$ does not satisfy (1) of Theorem 3.2.1 with k = 1 or 2.)

Indeed, applying (a) and (b) to $\mathcal{M} = \mathcal{S}(G,\Sigma)$ and $x = p$ yields the lemma. Statement (a) is straightforward to prove. Statement (b) is one of Seymour's "Splitter Theorems" (Seymour [1980]). $\square$

ALGORITHMIC CONSEQUENCES

Obviously, the decomposition results in this section yield polynomial-time algorithms to recognize whether or not a given signed graph contains no odd-K_4 and/or no odd-K_3^2. So, in particular we have a polynomial time algorithm to recognize whether or not a given bidirected graph has the Edmonds-Johnson property (cf. Corollary 2.3.10). The less obvious part of these algorithms is to recognize whether or not a given signed graph is planar with two odd faces. However this problem can be solved by using polynomial-time algorithms which give an embedding of the graph in the plane or decide that no such embedding exists. (For such algorithms cf. Auslander and Parter [1961], Hopcroft and Tarjan [1974].)

Clearly, the algorithms for recognizing signed graphs with no odd-K_4 and/or no odd-K_3^2 are special cases of algorithms for recognizing regular matroids based on Seymour's decomposition theorem (Theorem 3.2.1, Seymour [1980], cf. Cunningham and Edmonds [1980], Bixby, Cunningham, and Rajan [1986], Truemper [1987b]), and of algorithms for recognizing matroids containing no F_7-minor using some specific element (cf. Truemper and Tseng [1986], Truemper [1987a]).

3.3. ORIENTATIONS

An *orientation* of a signed graph is a replacement of the odd edges by directed edges. If an orientation is such that for each circuit the number of forwardly directed edges minus the number of backwardly directed edges is at most k in absolute, we say that the orientation has *discrepancy k*. (In counting these numbers we ignore the even edges in the circuit.) Obviously, a signed graph (G,Σ) has an orientation of discrepancy 0 if and only if (G,Σ) is bipartite.

<u>Theorem 3.3.1</u>
Let (G,Σ) *be a signed graph. Then* (G,Σ) *contains neither an odd-*K_4 *nor an odd-*K_3^2 *if and only if* (G,Σ) *has an orientation of discrepancy* 1.

<u>Proof</u>: The if part being trivial, we restrict ourselves to the only if part. For that assume that (G,Σ) contains no odd-K_4 an no odd-K_3^2. By Theorem 3.1.2(i), $\mathcal{S}(G,\Sigma)$ is regular. So, by Theorem 1.4.8 there exists a signing N of

$$\left[\begin{array}{c|c} 1 & x_\Sigma^T \\ \hline 0 & \\ \vdots & M_G \\ 0 & \end{array}\right] \quad \text{(cf. (3.1.1))}$$

which represents $\mathcal{S}(G,\Sigma)$ over $\mathbb{R}$. We may assume:

$$N = \left[\begin{array}{c|c} 1 & x_\Sigma^T \\ \hline 0 & \\ \vdots & \tilde{N} \\ 0 & \end{array}\right] \quad \text{with } \tilde{N} \equiv M_G \pmod 2).$$

(As we may multiply columns by -1.) Obviously, $\tilde{N}$ represents $\mathcal{M}(G)$ over $\mathbb{R}$.

Claim: *We may assume that each column of* $\tilde{N}$ *has one* 1 *and one* -1.

Proof of the Claim: Take a spanning forest F in G. By multiplying some of the rows of $\tilde{N}$ by -1, we can achieve that each column of $\tilde{N}$ corresponding to an edge in F contains one 1 and one -1. So the sum of the components of each of these columns is 0. But these columns span all the other columns of $\tilde{N}$ (as F is a basis of $\mathcal{M}(G)$). Hence, each column has one 1 and one -1.

end of proof of claim

Define the following orientation: Edge $e = uv \in \Sigma$ is directed from u to v if $\tilde{N}_{u,e} = -1$ (and so $\tilde{N}_{v,e} = 1$). We show that this orientation has discrepancy 1. Take a circuit C in G. Then $[\alpha_C | x_C^T]^T \in \mathcal{C}(\mathcal{S}(G,\Sigma))$, with $\alpha_C = 1$ if C is an odd circuit in (G,Σ) and $\alpha_C = 0$ if C is an even circuit in (G,Σ). By Theorem 1.4.7 there exists a signing $[\tilde{\alpha}_C, x_C^T]$ of $[\alpha_C, x_C^T]$ such that $\tilde{\alpha}_C + x_\Sigma^T x_C^T = 0$, and $\tilde{N}x_C = 0$. From this one easily derives that the number of forwardly directed edges minus the number of backwardly directed edges on C is $\pm\, \tilde{\alpha}_C$. This proves that the orientation constructed above has discrepancy 1. $\square$

Remark:

Theorem 3.3.1 can also be proved using Theorem 3.2.3 (and Lemma 3.2.2). We leave the details to the reader. The advantage of this alternative proof is that it provides a polynomial-time algorithm to find an orientation of discrepancy 1 on a signed graph with no odd-K_4 and no odd-K_3^2.

3.4. SHORTEST ODD CIRCUITS AND PACKING ODD CIRCUITS

If S is a finite set, $\mathcal{S}$ a collection of subsets of S, and $w \in \mathbb{Z}^S$, then a *w-packing* with elements of $\mathcal{S}$ is a family $S_1, S_2, \ldots, S_k$ of elements of $\mathcal{S}$ (repetition allowed) such that for each $s \in S$ we have that

$$|\{i=1,\ldots,k \mid s \in S_i\}| \leqq w_s.$$

The number k is called the *cardinality* of the family $S_1, \ldots, S_k$.

Seymour [1977] proved the following result.

<u>Theorem 3.4.1</u>

Let $\mathcal{M}$ be a binary matroid, and let $x \in E(\mathcal{M})$.
Then the following are equivalent:
(i) $\mathcal{M}$ does not contain an F_7-minor using x;
(ii) for each $w \in \mathbb{Z}_+^{E(\mathcal{M})}$:
$\min\{\sum_{e \in C \setminus \{x\}} w_e \mid C \cup \{x\}$ is a circuit of $\mathcal{M}\}$ is equal to the maximum cardinality of a w-packing with elements of $\{C^* \setminus \{x\} \mid C^* \cup \{x\}$ is co-circuit in $\mathcal{M}\}$. □

<u>Remark</u>:

For each binary matroid $\mathcal{M}$, with element x, the collection $\{C^* \setminus \{x\} \mid C^* \cup \{x\}$ is a co-circuit in $\mathcal{M}\}$, is exactly the collection of edge minimal sets meeting $C \setminus \{x\}$, for each circuit $C \cup \{x\}$ in $\mathcal{M}$. This proves that the maximum in Theorem 3.4.1 does not exceed the minimum in Theorem 3.4.1. (Needless to say that this is the easy part of the min-max relation, to prove "min $\leqq$ max" is the real job.)

Let (G,Σ) be a signed graph. Then a *Σ-boundary* of (G,Σ) is a subset of $E(G)$ of the form $\delta(U)\Delta\Sigma$ with $U \subset V(G)$. We denote the Σ-boundary $\delta(U)\Delta\Sigma$ by $[U]$ (e.g. $[\emptyset] = \Sigma$). The edge minimal Σ-boundaries are exactly the collection of subsets F of E(G) such that $F \cup \{p\}$ is a co-circuit of $\mathcal{S}(G,\Sigma)$.

Applying Theorem 3.4.1 to $\mathcal{S}(G,\Sigma)$ and $\mathcal{S}^*(G,\Sigma)$ we get (using Lemma 3.1.2(i) and Lemma 3.1.7(i)):

Theorem 3.4.2

Let (G,Σ) *be a signed graph*

(i) The following are equivalent:

- (G,Σ) *does not contain an odd-*K_4*;*
- *for each* $w\in\mathbb{Z}_+^{E(G)}$, *the maximum cardinality of a* w*-packing with odd circuits in* (G,Σ) *is equal to the minimum weight* $\sum_{e\in B} w_e$ *of a* Σ*-boundary* B *in* (G,Σ).

(ii) The following are equivalent:

- (G,Σ) *does not reduce to* $\tilde{K}_3^2$*;*
- *for each* $w\in\mathbb{Z}_+^{E(G)}$ *the minimum weight* $\sum_{e\in E(C)} w_e$ *of an odd circuit* C *in* (G,Σ) *is equal to the maximum cardinality of a* w*-packing with* Σ*-boundaries in* (G,Σ). □

Corollary 3.4.3

Let (G,Σ) *be a signed graph. Then* (G,Σ) *contains neither an odd-*K_4 *nor an odd-*K_3^2 *if and only if for each* $w\in\mathbb{Z}_+^{E(G)}$ *both min-max relations in Theorem 3.4.2 hold.* □

In this section we use the orientation Theorem 3.3.1 to give an alternative proof of Corollary 3.4.3.

Remarks:

Using Theorem 3.2.5 and Theorem 3.2.7 one can derive Theorem 3.4.2 as a corollary of Corollary 3.4.3. We skip this derivation, as the techniques are similar to the techniques used in the proof of Theorem 4.3.2. Our purpose is to show how the min-max relations in Theorem 3.4.2 can be formulated as min-max relations for certain min-cost flow problems, in case (G,Σ) contains no odd-K_4 and no odd-K_3^2.

The derivation of Theorem 3.4.2 from Corollary 3.2.3, using Theorem 3.2.5 and Theorem 3.2.6 can be viewed as a special instance of Truemper's derivation of Theorem 3.4.1 from the special case of Theorem 3.4.1 where $\mathcal{M}$ is regular (Truemper [1987a], he uses a strengthed form of the decomposition theorem for binary matroids with no F_7-minor using some specific element due to Tseng and Truemper [1986]). The proof of Corollary 3.4.3 can be viewed as a special instance of the proof of Theorem 3.4.1 for the special case that $\mathcal{U}$ is regular (for such proofs cf. Gallai [1959b], Minty [1966], Fulkerson [1968]).

PACKING ODD CIRCUITS

Let (G,Σ) be a signed graph. Moreover let $w \in \mathbb{Z}_+^{E(G)}$. The *odd circuit packing problem* in (G,Σ) is

(3.4.4) *Find a maximum cardinality* w*-packing with odd circuits in* (G,Σ).

The *shortest* Σ*-boundary problem* in (G,Σ) is

(3.4.5) *Find a* Σ*-boundary* $[U]$ *in* (G,Σ) *(with* $U \subseteq V(G)$*), such that* $\sum_{e \in [U]} w_e$ *is minimal.*

From now on, assume that (G,Σ) has no odd-K_4 and no odd-K_3^2. So, by Theorem 3.1.1, (G,Σ) has an orientation of discrepancy 1. Let $\vec{A}$ be the set of arcs in such orientation, together with, for each even edge, uv, in (G,Σ) an arbitrarily directed arc, $\overrightarrow{uv}$ or $\overrightarrow{vu}$. For each arc $\overrightarrow{uv} \in \vec{A}$ we add a new arc $\overrightarrow{vu}$, $\overleftarrow{A} := \{\overrightarrow{vu} \mid \overrightarrow{uv} \in \vec{A}\}$. Consider the following circulation problem.

(3.4.6) $\max \sum_{a \in \vec{A} \cap \Sigma} f_a - \sum_{a \in \overleftarrow{A} \cap \Sigma} f_a$

s.t. f is a nonnegative circulation in $(V, \vec{A} \cup \overleftarrow{A})$,
such that for each $a_1 \in \overleftarrow{A}$, $a_2 \in \vec{A}$ coming from the same edge
$e \in E(G)$: $f_{a_1} + f_{a_2} \leqq w_e$.

(Here $a \in \vec{A} \cap \Sigma$ ($\overleftarrow{A} \cap \Sigma$) means that $a \in \vec{A}$ ($\overleftarrow{A}$ respectively) and comes from an odd edge.)

The linear programming dual of (3.4.6) is

(3.4.7) $\min \sum_{e \in E(G)} w_e \delta_e$

s.t. $\delta_e \in \mathbb{Q}_+^{E(G)}$, with the property there exists a $\pi \in \mathbb{Q}^{V(G)}$ satisfying for each $\overrightarrow{uv} \in \vec{A}$:

$$1 - \delta_{uv} \leqq \pi_v - \pi_u \leqq 1 + \delta_{uv} \quad \text{if } uv \in \Sigma;$$
$$- \delta_{uv} \leqq \pi_v - \pi_u \leqq \delta_{uv} \quad \text{if } uv \notin \Sigma.$$

Remark:

Formulated as it is, (3.4.6) is not a proper circulation problem. However it can be transformed into a circulation problem as follows: replace each pair $a_1 \in \vec{A}$, $a_2 \in \overleftarrow{A}$, coming from one edge $e = uv \in E(G)$ by the configuration in Figure 3.11. To arc $\vec{e}$ we assign a capacity w_e, while all other new capacities are ∞.

Figure 3.11

Proposition 3.4.8: *The maximum in (3.4.6) is attained by an integer vector $f \in \mathbb{Z}^{\vec{A} \cup \overleftarrow{A}}$.*

Proof: By the remark above. □

Proposition 3.4.9: *(3.4.4) and (3.4.6) are equivalent.*

Proof: For each circuit C in (G,Σ) we define a circulation f^C as follows. In $\vec{A} \cup \overleftarrow{A}$ there are two directed circuits corresponding in a natural way with C. Select one of these two circuits, such that the selected circuit uses at least as many arcs from $\vec{A}$ as from $\overleftarrow{A}$. Call the selected circuit $\vec{C}$. Now $f^C \in \{0,1\}^{\vec{A} \cup \overleftarrow{A}}$ is defined by $f^C_a = 1$ if and only if $a \in \vec{C}$.

Let $C_1, \ldots, C_t$ be a w-packing by odd circuits. Then $f^{C_1} + \ldots + f^{C_t}$ is a feasible solution of (3.4.6) with objective value:

$$\sum_{i=1}^{t}\left[\sum_{a\in\vec{A}\cap\Sigma} f_a^{C_i} - \sum_{a\in\overleftarrow{A}\cap\Sigma} f_a^{C_i}\right] =$$

$$\sum_{i=1}^{t}\left(|\vec{A}\cap\Sigma\cap A(\vec{C}_i)| - |\overleftarrow{A}\cap\Sigma\cap A(\vec{C}_i)|\right) = t,$$

as $\vec{A}$ is an orientation of discrepancy 1. ($A(\vec{C}_i)$ denotes the set of arcs in $\vec{A}\cup\overleftarrow{A}$ belonging to $\vec{C}_i$.)

Conversely, let f be an integer valued feasible solution of (3.4.6). Obviously there exist circuits $C_1,\ldots,C_2$ in G such that $f = f^{C_1} + \ldots + f^{C_f}$. The number of odd circuits among $C_1,\ldots,C_t$ is at least the objective value of f. Since f is feasible to (3.4.6), these odd circuits form a w-packing. Proposition 3.4.9 now follows from the above combined with Proposition 3.4.8. □

Define, for each $\delta\in\mathbb{Q}_+^{E(G)}$, the weight function $\vec{\delta}\in\mathbb{Q}^{\vec{A}\cup\overleftarrow{A}}$ by:

$$\vec{\delta}_a = \begin{cases} \delta_e+1 & \text{if } a\in\vec{A} \text{ and a comes from } e\in\Sigma; \\ \delta_e-1 & \text{if } a\in\overleftarrow{A} \text{ and a comes from } e\in\Sigma; \\ \delta_e & \text{else.} \end{cases}$$

Using Theorem 1.3.2 we can reformulate (3.4.7) as:

(3.4.10) $\min \sum_{e\in E(G)} w_e\delta_e$

s.t. $\delta\in\mathbb{Q}_+^{E(G)}$, with the property that there exists no directed circuit $\vec{C}$ in $\vec{A}\cup\overleftarrow{A}$, with $\sum_{a\in A(\vec{C})} \vec{\delta}_a < 0$.

<u>Proposition 3.4.11</u>: *(3.4.7) has an optimal solution* $(\delta,\pi)\in\{0,1\}^{E(G)}\times\mathbb{Z}^{V(G)}$.

<u>Proof</u>: From Proposition 3.4.8 and Corollary 1.2.19 it follows that (3.4.7) has an optimal solution $(\delta,\pi)\in\mathbb{Z}_+^{E(G)}\times\mathbb{Q}^{V(G)}$. From this one easily sees that also π can assumed to be integer valued (e.g. using Theorem 1.3.2).

Now let (δ,π) be an integer valued optimal solution of (3.4.7), with $\sum_{e\in E(G)} \delta_e$ as small as possible. Let $e^*\in E(G)$. Define $\delta^*\in \mathbb{Z}^{E(G)}$ by $\delta^*_{e^*} = \delta_{e^*}-1$, and $\delta^*_e = \delta_e$ if $e\neq e^*$. Then δ^* is not feasible for 3.4.10. Hence $\delta^*_{e^*} < 0$, i.e. $\delta_{e^*} = 0$, or there exists a directed circuit $\vec{C}$, in $\vec{A}\cup\overleftarrow{A}$ such that $\sum_{a\in A(\vec{C})} \vec{\delta}_a = 0$. Since $\vec{A}$ has discrepancy 1, this means that

$$\delta_{e^*} \leqq \sum_{e\in A(C)} \delta_e - \sum_{a\in A(\vec{C})} \vec{\delta}_a \in\{0,\pm1\}.$$

Hence $\delta_{e^*} \leqq 1$, and it follows that $\delta\in\{0,1\}^{E(G)}$. □

From Proposition (3.4.11) it follows that

(3.4.12) $\min(3.4.7) \geqq \min(3.4.5)$.

Indeed, let $(\delta,\pi)\in\{0,1\}^{E(G)}\times\mathbb{Z}^{V(G)}$ be an optimal solution to (3.4.7). Define $V := \{u\in V(G)\,|\,\pi_u \text{ even}\}$. It is straightforward to check that $\delta_e = 1$ if and only if $e\in[V]$. So $[V]$ is a Σ-boundary with $\sum_{e\in[V]} w_e = \sum_{e\in E(G)} w_e\delta_e$.

Using (3.4.12), linear programming duality and Proposition (3.4.9) we get:

$$\min(3.4.5) \leqq \min(3.4.7) = \max(3.4.6) = \max(3.4.4) \leqq \min(3.4.5).$$

So we have the following

<u>Conclusion</u>:

If (G,Σ) is a signed graph with no odd-K_4 and no odd-K_3^2, then the $\min(3.4.5) = \max(3.4.4)$ for each $w\in\mathbb{Z}_+^{E(G)}$.

SHORTEST ODD CIRCUIT

Let (G,Σ) be a signed graph. Moreover let $w\in\mathbb{Z}_+^{E(G)}$. The *shortest odd circuit problem* in (G,Σ) is:

(3.4.13) *Find an odd circuit C in (G,Σ) which minimizes* $\sum_{e\in E(C)} w_e$.

The *packing with Σ-boundaries problem* in (G,Σ) is:

(3.4.14) *Find a maximum cardinality w-packing of Σ-boundaries in* (G,Σ).

From now on we assume that (G,Σ) has no odd-K_4 and no odd-K_3^2. So, by Theorem 3.1.1, (G,Σ) has an orientation of discrepancy 1. Let $\vec{A}$ en $\overleftarrow{A}$ be defined as before (cf. the paragraph following (3.4.5)).

Define for each $\sigma \geqq 0$ the following weight function $w^{\sigma} \in \mathbb{Q}^{\vec{A} \cup \overleftarrow{A}}$ by

$$w_a^{\sigma} := \begin{cases} w_e - \sigma & \text{if } a \in \vec{A}, \text{ and a comes from } e \in \Sigma; \\ w_e + \sigma & \text{if } a \in \overleftarrow{A}, \text{ and a comes from } e \in \Sigma; \\ w_e & \text{else.} \end{cases}$$

From the fact that $\vec{A}$ has discrepancy 1 it follows that for each $\sigma \in \mathbb{Q}$ and each directed circuit $\vec{C}$ in $\vec{A} \cup \overleftarrow{A}$ (coming from circuit C in G) the following holds:

$$\sum_{a \in A(\vec{C})} w_a^{\sigma} - \sum_{e \in E(C)} w_e = \begin{cases} \pm\,\sigma & \text{if C is odd;} \\ 0 & \text{if C is even.} \end{cases}$$

From this we see that (3.4.13) can be reformulated as:

(3.4.15) max σ

s.t. $\sigma \in \mathbb{Q}$, with the property that there exists no directed circuit $\vec{C}$ in $\vec{A} \cup \overleftarrow{A}$ for which $\sum_{a \in A(\vec{C})} w_a^{\sigma} < 0$.

This, in term, is equivalent to: (cf. Theorem 1.3.2)

(3.4.16) max σ

s.t. $\sigma \in \mathbb{Q}$, with the property that there exists an $\pi \in \mathbb{Q}^{V(G)}$ such that for each $\overrightarrow{uv} \in \vec{A}$:

$$|\pi_v - \pi_u + \sigma| \leqq w_{uv} \quad \text{if } uv \in \Sigma;$$
$$|\pi_v - \pi_u| \leqq w_{uv} \quad \text{if } uv \notin \Sigma.$$

Let σ^* be the length of the shortest odd circuit in (G,Σ) (with respect to w). As $w\in\mathbb{Z}^{E(G)}$, σ^* is integral. Since min (3.4.13) = max (3.4.16), there exists a $\pi^*\in\mathbb{Q}^{V(G)}$ such that (σ^*,π^*) is an optimal solution of (3.4.16). By Theorem 1.3.2 we may assume that π^* is integer valued. (In fact, for $a\in V(G)$, we can take π^*_u as the minimum weight, with respect to w^{σ^*}, of any directed path in $\vec{A}\cup\overleftarrow{A}$ with endpoint u.)

Now we shall construct a w-packing of Σ-boundaries with cardinality σ^* as follows:

$$\begin{aligned}&\text{For each } i=1,\dots,\sigma^*,\\ &\qquad Z_i:=\{z\in\mathbb{Z}\mid z=i+1,\ i+2,\dots,i+\sigma^* \pmod{2\sigma^*}\}\\ &\text{and}\quad V_i:=\{u\in V(G)\mid \pi^*_u\in Z_i\}.\end{aligned}$$

Then $[V_1],\dots,[V_{\sigma^*}]$ is a w-packing. Indeed, this follows easily from the following three:

(i) $uv\in[V_i]\cap\Sigma$ if and only if $|\{\pi^*_u,\ \pi^*_v+\sigma^*\}\cap Z_i| = 1$;
(ii) $uv\in[V_i]\setminus\Sigma$ if and only if $|\{\pi^*_u,\ \pi^*_v\}\cap Z_i| = 1$;
(iii) for $z_1, z_2\in\mathbb{Z}$:
$$|\{i=1,\dots,\sigma^*\mid\ |\{z_1,z_2\}\cap Z_i| = 1\}| \leqq \min\{|z_1-z_2|,\sigma^*\}.$$

So, we have the following

<u>Conclusion</u>:
If (G,Σ) is a signed graph with no odd-K_4 and no odd-K_3^2, then min (3.4.13) = max (3.4.14) for each $w\in\mathbb{Z}_+^{E(G)}$.

<u>Remarks</u>:
(i) There exist polynomial-time algorithms which find a minimum weight odd circuit (in any signed graph, cf. Grötschel and Pulleyblank [1981], Gerards and Schrijver [1986]). For signed graphs with no odd-K_4 and no odd-K_3^2 the discussion above yields an easy polynomial-time algorithm for solving the packing with Σ-boundaries problem at least as soon as the orientation with discrepancy 1 is known (cf. final remark of Section 3.3). Indeed, first we find the minimum weight, σ^* say of an odd circuit in (G,Σ). Then we calculate for each $u\in V(G)$, π^*_u

as the length, with respect to w^{σ^*}, of the shortest directed path in $\vec{A}\cup\overleftarrow{A}$ with endpoint u. Now we find a w-packing of Σ-boundaries as follows: (Note that we have to be careful since σ* can be exponential in the size of the problem.)

- $D := \{d \mid 0 \leqq d \leqq \sigma^*$, there exists a $u \in V(G)$ with $\pi^*_u \equiv d(\text{mod } \sigma^*)\}$.
- Assume $d_1 < d_2 < \ldots < d_k$ such that $D = \{d_1,\ldots,d_k\}$.
- Let $\lambda_i := d_i - d_{i-1}$ for $i = 2,\ldots,k$, and $\lambda_1 := d_1 - d_k + \sigma^*$.
- Taking each Σ-boundary $[V_{d_i}]$ with multiplicity λ_i $(i=1,\ldots,k)$, we get a w-packing of Σ-boundaries. The cardinality of this packing is $\sum_{d\in D} \lambda_d = \sigma^*$, as is easily verified.

(ii) We can reformulate the shortest odd circuit problem in signed graphs with no odd-K_4 and no odd-K_3^2 as

$$(3.4.17)\quad \max \sum_{a\in\vec{A}\cup\overleftarrow{A}} w_a f_a$$
$$\text{s.t. } f \text{ is a non-negative circulation in } \vec{A}\cup\overleftarrow{A} \text{ such that}$$
$$\sum_{a\in\vec{A}\cap\Sigma} f_a - \sum_{a\in\overleftarrow{A}\cap\Sigma} f_a = 1.$$

(3.4.17) is the dual of (3.4.16). One easily proves that (3.4.17) has an integer valued optimal solution.

3.5. HOMOMORPHISMS TO ODD CIRCUITS AND 3-COLOURABILITY

In this section we prove two graph theoretic results, one due to Catlin [1979], the other to Gerards [1987]. We start with the latter result.

HOMOMORPHISMS TO ODD CIRCUITS

Let G_1 and G_2 be two undirected graphs. We call a map φ: $V(G_1) \to V(G_2)$ a *homomorphism* from G_1 to G_2, if $\varphi(u_1)\varphi(u_2) \in E(G_2)$ for each $uv \in G_1$. A *parity preserving subdivision* of a signed graph (G,Σ) is an undirected graph, obtained from G by replacing each odd (even) edge in G by a path of odd (even) length. The following result is another characterization of signed graphs with no odd-K_4 and no odd-K_3^2.

Theorem 3.5.1 (Gerards [1987])
Let (G,Σ) be a signed graph. Then (G,Σ) contains no odd-K_4 and no odd-K_3^2 if and only if for each parity preserving subdivision G^1 of G, there exists a homomorphism from G^1 to the shortest circuit in G^1.

Proof: We leave the if part to the reader. (E.g. for the graphs in Figure 3.12(a), (b) there exists no homomorphism to their shortest odd circuit. However, for the graph in Figure 3.12(c) such a homomorphism exists!)

To prove the only if part, let (G,Σ) be a signed graph with no odd-K_4 and no odd-K_3^2. Let G' be a parity preserving subdivision of (G,Σ). With no loss of generality we may assume that $\Sigma = E(G)$, and $G' = G$. By Theorem 3.3.1, G has an orientation, $\vec{A}$ say, of discrepancy 1. Let $\overleftarrow{A} := \{\overrightarrow{uv} \mid \overrightarrow{vu} \in \vec{A}\}$. Assume the length of the shortest odd circuit is $2k + 1$. Define $w \in \mathbb{Z}^{\vec{A}\cup\overleftarrow{A}}$ by:

$$w_a := \begin{cases} k + 1 & \text{if } a \in \vec{A}, \\ -k & \text{if } a \in \overleftarrow{A}. \end{cases}$$

As $\vec{A}$ has discrepancy 1, $\vec{A}\cup\overleftarrow{A}$ has no directed circuit with negative weight with respect to w. So, by Theorem 1.3.2, there exists a $\varphi \in \mathbb{Z}^{V(G)}$ satisfying:

$$\varphi_u - \varphi_v \leqq w_{\overrightarrow{uv}} \text{ if } \overrightarrow{uv} \in \vec{A} \cup \overleftarrow{A}.$$

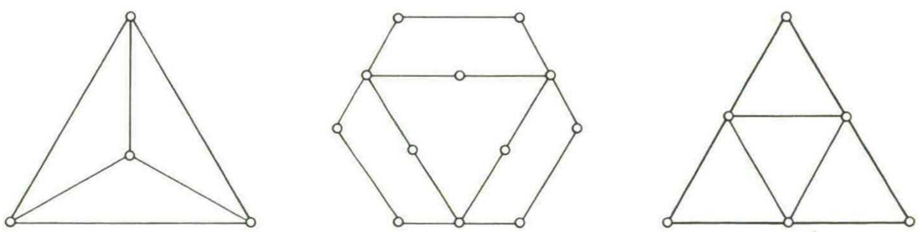
So φ satisfies:

$$k \leqq \varphi_u - \varphi_v \leqq k + 1 \text{ if } \overrightarrow{uv} \in \vec{A}.$$

Hence:

$$2\varphi_u - 2\varphi_v = \pm 1 \pmod{2k+1} \text{ if } uv \in E(G).$$

So $u \to 2\varphi_u \pmod{2k+1}$ maps G to a circuit of length 2k+1. □



(a) (b) (c)

Figure 3.12

Remarks:

(i) The proof above is due to A. Schrijver. It relies on Theorem 3.3.1, and hence on Tutte's characterization of regular matroids (Theorem 1.4.4). A direct and elementary, though more complicated, proof of Theorem 3.5.1 can be found in Gerards [1987].

(ii) Schrijver observed that Theorem 3.5.1 can be used to prove the min-max relation of Theorem 3.4.2(ii) for signed graphs (G,Σ) with no odd-K_4 and no odd-K_3^2 and weight functions w which satisfy: $\{e|w_e \text{ is odd}\} = \Sigma$.

(iii) Theorem 3.5.1 extends a result of Albertson, Catlin, and Gibbons [1985] stating that an undirected graph G can be mapped homomorphically to an odd circuit of length M if no subgraph of G can be fold to a homeomorph of K_4 in which all triangles of K_4 have become circuits of length M (*fold* means repeatedly identifying nodes at distance two). Related results can be found in Catlin [1984] and Lai [1987].

3-COLOURABILITY

The other graph-theoretic result we want to mention in this section is:

Theorem 3.5.2 (Catlin [1979])
Let G *be an undirected graph, such that* (G,E(G)) *has no odd-*K_4. *Then* G *is* 3-*colourable.*

Proof: Let G be a minimal counterexample. If (G,E(G)) contains no odd-K_3^2, then there exists a homomorphism of G to its shortest odd circuit, so certainly to K_3. This implies that G is 3-colourable. So (G,E(G)) has an odd-K_3^2. Hence, by Theorem 3.2.5, G has a two node cutset. (Obviously $(G,E(G)) \not\simeq \tilde{K}_3^2$.) Now, one side of this two node cutset (possibly after adding an edge between the two nodes in the cutset) is a smaller counterexample. $\square$

Remark:
The technique used in the proof of Theorem 3.5.1 is similar to the technique Minty used to prove the following result:

A graph G *is* k-*colourable if and only if* G *has an orientation such that for each circuit* C *the number of forwardly directed arcs is at least* $\frac{1}{k}|E(C)|$ *(Minty [1962]).*

3.6. AN EXTENSION OF KÖNIG'S THEOREM TO GRAPHS WITH NO ODD-K_4

Throughout this section $G = (V(G),E(G))$ denotes an undirected graph without isolated nodes. Each time we use some notions from signed graphs, e.g. odd-K_4 and odd-K_3^2, we implicitly consider the signed graph $(G,E(G))$; so we consider all edges to be odd.

In this section we give an extension of the following well-known result.

(3.6.1) *If* G *has no odd circuit,*

then $\alpha(G) = \rho(G)$ *and* $\tau(G) = \nu(G)$ (König [1931, 1933]).

As usual, the parameters α, ρ, τ and ν are defined as:

$\alpha(G)$:= the maximum cardinality of a stable set in G. ($S \subset V(G)$ is a *stable set* if $u,v \in S$ implies $uv \notin E(G)$.)

$\rho(G)$:= the minimum cardinality of an edge-cover for G. ($E' \subset E(G)$ is an *edge-cover* if for each $u \in V$ there exists an $e \in E'$ with endpoint u.)

$\nu(G)$:= the maximum cardinality of a matching in G. ($M \subset E(G)$ is a *matching* if e_1, $e_2 \in M$, $e_1 \neq e_2$ implies e_1 and e_2 have no common endpoint.)

$\tau(G)$:= the minimum cardinality of a node-cover for G. ($N \subset V(G)$ is a *node-cover* if $uv \in E(G)$ implies $u \in N$ or $v \in N$.)

We introduce two new parameters:

$\tilde{\rho}(G)$:= the minimum cost of a collection of edges and odd circuits in G covering the nodes of G. The *cost of an edge* is equal to 1, and the *cost of a circuit* with 2k+1 edges is equal to k. The *cost of a collection of edges and odd circuits* is equal to the sum of the costs of its members.

$\tilde{\nu}(G)$:= the maximum profit of a collection of mutually node disjoint edges and odd circuits in G. The *profit of an edge* is equal to 1 and the *profit of a circuit* of length 2k+1 is equal to k+1. The *profit of a collection of edges and odd circuits* is equal to the sum of the profits of its members.

The following inequalities are obvious:

$$\text{(3.6.2)}\qquad \begin{aligned} &\alpha(G) \leqq \tilde{\rho}(G) \leqq \rho(G), \\ &\tau(G) \geqq \tilde{\nu}(G) \geqq \nu(G). \end{aligned}$$

König's Theorem (3.6.1) can be extended to the following result. (It follows from the more general Theorem 3.6.8 stated below.)

<u>Theorem 3.6.3</u>

Let G *be an undirected graph, without isolated nodes. If* G *does not contain an odd-*K_4 *as a subgraph, then* $\alpha(G) = \tilde{\rho}(G)$ *and* $\tau(G) = \tilde{\nu}(G)$. □

To see that Theorem 3.6.3 extends König's Theorem (3.6.1), observe that a bipartite graph G has no odd-K_4, and trivially satisfies $\tilde{\rho}(G) = \rho(G)$, $\tilde{\tau}(G) = \tau(G)$ (as G has no odd circuits).

The two equalities in (3.6.1) are equivalent, for any graph G. This follows from

(3.6.4) $\alpha(G) + \tau(G) = |V(G)| = \rho(G) + \nu(G)$ (Gallai [1958, 1959a]).

A similar equivalence for the equalities $\alpha(G) = \tilde{\rho}(G)$ and $\tau(G) = \tilde{\nu}(G)$ follows from the following result of Schrijver [personal communication], analogous to Gallai's result above.

<u>Theorem 3.6.5</u>

Let G *be an undirected graph without isolated nodes. Then* $\tilde{\rho}(G) + \tilde{\nu}(G) = |V(G)|$.

<u>Proof</u>: First, let $e_1,\dots,e_m$, $C_1,\dots,C_n$ be a collection of mutually node disjoint edges and odd circuits such that the profit $m + \sum_{i=1}^{n} \frac{1}{2}(|V(C_i)| + 1)$ of the collection is equal to $\tilde{\nu}(G)$.

Let $V_1 := V(G)\setminus \bigcup_{i=1}^{n} V(C_i)$, and let G_1 be the subgraph of G induced by V_1. Then obviously $m = \nu(G_1)$. Let $f_1,\dots,f_{\rho(G_1)}$ be a minimum edge-cover for G_1. Then $f_1,\dots,f_{\rho(G_1)}$, $C_1,\dots,C_n$ is a collection of edges and odd circuits covering V(G). The cost of this collection is (using Gallai's identity (3.6.4)):

$$\rho(G_1) + \sum_{i=1}^{n} \tfrac{1}{2}(|V(C_i)|- 1) = |V_1| - \nu(G_1) - \sum_{i=1}^{n} \tfrac{1}{2}(|V(C_i)| + 1) +$$

$$\sum_{i=1}^{n} |V(C_i)| = |V(G)| - \tilde{\nu}(G).$$

Hence $\tilde{\rho}(G) + \tilde{\nu}(G) \leqq |V(G)|$.

The reverse inequality is proved almost identically. However there is a small technical difficulty, settled in the claim below.

Let $e_1,\dots,e_m$, $C_1,\dots,C_n$ be a collection of edges and odd circuits covering V(G) such that the cost $m + \sum_{i=1}^{n} \frac{1}{2}(|V(C_i)| - 1)$ of the collection is equal to $\tilde{\rho}(G)$, and such that, moreover, n is small as possible.

<u>Claim</u>: For each $i,j=1,\dots,n$ $(i\neq j)$; $k=1,\dots,m$ we have $V(C_i)\cap V(C_j) = \emptyset$, and no endpoint of e_k is element of $V(C_i)$.
<u>Proof of Claim</u>: Suppose $u\in V(C_i)$ $(i=1,\dots,n)$, such that u is also contained in another odd circuit among $C_1,\dots,C_n$, or in one of the edges $e_1,\dots,e_m$. Let $f_1,\dots,f_p \in E(C_i)$ be the unique maximum cardinality matching in C_i not covering u. Then $p = \frac{1}{2}(|V(C_i)|- 1)$. Obviously $e_1,\dots,e_m$, $f_1,\dots,f_p$, $C_1,\dots,C_{i-1}$, $C_{i+1},\dots,C_n$ is a collection of edges and odd circuits covering V(G). Its cost is $\tilde{\rho}(G)$. However it contains only n-1 odd circuits, contradicting the minimality of n.

<u>end of proof of claim</u>.

As before we define $V_1 = V(G)\setminus \bigcup_{i=1}^{n} V(C_i)$ and G_1 as the subgraph of G induced by V_1. By similar arguments as used in the first part of the proof one gets:

$$\begin{aligned}\tilde{\rho}(G) &= \rho(G_1) + \sum_{i=1}^{n} \tfrac{1}{2}(|V(C_i)|- 1)\\ &= |V_1| - \nu(G_1) - \sum_{i=1}^{n} \tfrac{1}{2}(|V(C_i)|+ 1) + \sum_{i=1}^{n} |V(C_i)|\\ &\geqq |V(G)| - \tilde{\nu}(G).\end{aligned}$$ □

<u>Corollary 3.6.6</u>
Let G be an undirected graph without isolated nodes. Then $\alpha(G) = \tilde{\rho}(G)$ *if and only if* $\tau(G) = \tilde{\nu}(G)$. □

As mentioned before we prove a more general weighted version of Theorem 3.6.3 (Theorem 3.6.8 below).

WEIGHTED VERSIONS

We define weighted versions of the numbers α, ρ, ν, τ, $\tilde{\rho}$, and $\tilde{\nu}$ and state the obvious generalizations of the results mentioned.
Let $w \in \mathbb{Z}^{V(G)}$.

$\alpha_w(G)$:= maximum $\{\sum_{u\in S} w_u | S$ is a stable set in G$\}$.

$\rho_w(G)$:= the minimum cardinality of a w-edge-cover for G. (A *w-edge-cover* for G is a collection $e_1,\ldots,e_m$ in E(G)) (repetition allowed) such that for each $u\in V(G)$ there are at least w_u edges among $e_1,\ldots,e_m$ incident with u. The *cardinality* of $e_1,\ldots,e_m$ is m.)

$\nu_w(G)$:= the maximum cardinality of a w-matching in G. (A *w-matching* is a collection $e_1,\ldots,e_m$ in E(G) (repetition allowed) such that for each $u\in E(G)$ there are at most w_u edges among $e_1,\ldots,e_m$ incident with u.)

$\tau_w(G)$:= minimum $\{\sum_{u\in N} w_u | N$ is node-cover for G$\}$.

Moreover we define:

A w-*cover* (w-*packing*, respectively) *by edges and odd circuits* is a collection $e_1,\dots,e_m$ of edges and $C_1,\dots,C_m$ of odd circuits (repetition allowed), such that for each $u\in V(G)$:

$$|\{i=1,\dots,m \mid u \text{ endpoint of } e_i\}| + |\{i=1,\dots,n \mid u\in V(C_i)\}| \geqq w_u$$
$$(\leqq w_u \text{ respectively}).$$

The *cost* of $e_1,\dots,e_m$, $C_1,\dots,C_n$ is $m + \sum_{i=1}^{n} \frac{1}{2}(|V(C_i)| - 1)$, its *profit* is $m + \sum_{i=1}^{n} \frac{1}{2}(|V(C_i)| + 1)$.

$\tilde{\rho}_w(G)$:= the minimum cost of a w-cover by edges and odd circuits in G.

$\tilde{\nu}_w(G)$:= the maximum profit of a w-packing by edges and odd circuits in G.

Remark:

The notion of "w-packing" is defined in Section 3.4. To bring the definition above in line with the definition in Section 3.4 define $S := V(G)$, and $\mathcal{S} := \{\{u,v\} \mid uv\in E(G)\} \cup \{V(C) \mid C \text{ odd circuit}\}$. Note however that the cardinality of a w-packing defined in Section 3.4 is not the same as the profit of a w-packing.

The numbers defined above satisfy:

(3.6.7) *If G has no odd circuit, then $\alpha_w(G) = \rho_w(G)$ and $\tau_w(G) = \nu_w(G)$* (Egerváry [1931]),

$$\alpha_w(G) \leqq \tilde{\rho}_w(G) \leqq \rho_w(G),$$
$$\tau_w(G) \geqq \tilde{\nu}_w(G) \geqq \nu_w(G),$$
$$\alpha_w(G) + \tau_w(G) = \tilde{\rho}_w(G) + \tilde{\nu}_w(G) = \rho_w(G) + \nu_w(G) = \sum_{u\in V(G)} w_u.$$

(3.6.7) can be proved easily from the cardinality versions stated before (with $w \equiv 1$), using the following construction. Define G_w by:

$$V(G_w) = \{[u,i] \mid u\in V(G);\ i=1,\dots,w_u\},$$

$$E(G_w) = \{[u,i][v,j] \mid u,v\in V(G);\ uv\in E(G);\ i=1,\ldots,w_u;\ j=1,\ldots,w_v\}.$$

Then one easily proves that $\alpha_w(G) = \alpha(G_w)$, $\rho_w(G) = \rho(G_w)$, $\nu_w(G) = \nu(G_w)$, $\tau_w(G) = \tau(G_w)$, $\tilde{\rho}_w(G) = \tilde{\rho}(G_w)$, $\tilde{\nu}_w(G) = \tilde{\nu}(G_w)$, and $V(G_w) = \sum_{u\in V(G)} w_u$. Moreover G_w is bipartite if and only if G is. All this yields (3.6.7). Theorem (3.6.3) can be generalized as well:

<u>Theorem 3.6.8</u>

Let G be an undirected graph, without isolated nodes. If G contains no odd-K_4 as a subgraph, then $\alpha_w(G) = \tilde{\rho}_w(G)$ and $\tau_w(G) = \tilde{\nu}_w(G)$ for any $w\in\mathbb{Z}^{V(G)}$. □

We prove this theorem later in this section. It should be noted that Theorem 3.6.8 does not follow from Theorem 3.6.3 by using G_w. The reason is that it is possible that G_w contains an odd-K_4 even if G does not. This is illustrated by the graph in Figure 3.13. (The bold edges, in Figure 3.13b form an odd-K_4.)

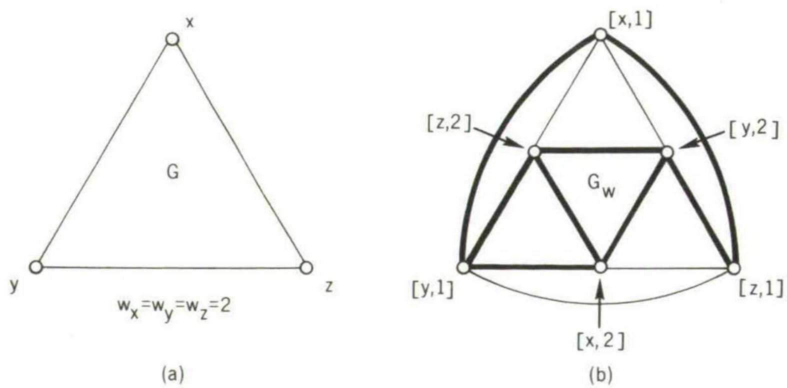


Figure 3.13

The statement "$\alpha_w(G) = \tilde{\rho}_w(G)$ for each $w\in\mathbb{Z}^{V(G)}$" can be reformulated in terms of integer linear programming.

(3.6.9) Both optima in the following primal-dual pair of linear programs, are attained by integral vectors if w is integer valued.

PRIMAL:

$$\begin{aligned}
&\max \sum_{u\in V(G)} w_u x_u && \\
&\text{s.t. } x_u + x_v \leqq 1 && (uv\in E(G));\\
&\qquad \sum_{u\in V(C)} x_u \leqq \tfrac{1}{2}(|V(C)| - 1) && (C\in\Gamma(G));\\
&\qquad\qquad x_u \geqq 0 && (u\in V(G)).
\end{aligned}$$

DUAL:

$$\begin{aligned}
\tilde{\rho}^*_w(G) := &\min \sum_{e\in E(G)} y_e + \sum_{C\in\Gamma(G)} \tfrac{1}{2}(|V(C)| - 1)z_C && \\
&\text{s.t. } \sum_{\substack{e\in E(G)\\ e\ni u}} y_e + \sum_{\substack{C\in\Gamma(G)\\ u\in V(C)}} z_C \geqq w_u && (u\in V(G));\\
&\qquad\qquad y_e \geqq 0 && (e\in E(G));\\
&\qquad\qquad z_C \geqq 0 && (C\in\Gamma(G)).
\end{aligned}$$

($\Gamma(G)$ denotes the collection of odd circuits $C = (V(C), E(C))$ in G.)

Before proving Theorem 3.6.8, we prove a special case:

Theorem 3.6.10
Let G be an undirected graph without isolated nodes. If G contains neither an odd-K_4 nor an odd-K_3^2, then $\alpha_w(G) = \tilde{\rho}_w(G)$ and $\tau_w(G) = \tilde{\nu}_w(G)$ for each $w\in\mathbb{Z}^{V(G)}$.

Proof: According to Theorem 3.3.1, G has an orientation with discrepancy 1. Let $\vec{A}$ denote the set of arcs in this orientation. For each $\overrightarrow{uv}\in\vec{A}$ we add a reversely directed arc $\overrightarrow{vu}$ too. Denote $\overleftarrow{A} := \{\overrightarrow{vu} \mid \overrightarrow{uv}\in\vec{A}\}$. Consider the following "circulation" problem:

$$(3.6.11)\quad \min \sum_{a\in\vec{A}} f_a$$

$$\text{s.t.} \sum_{\substack{a\in\vec{A}\cup\overleftarrow{A} \\ a \text{ enters } u}} f_a - \sum_{\substack{a\in\vec{A}\cup\overleftarrow{A} \\ a \text{ leaves } u}} f_a = 0 \qquad (u\in V(G));$$

$$\sum_{\substack{a\in\vec{A}\cup\overleftarrow{A} \\ a \text{ enters } u}} f_a \geqq w_u \qquad (u\in V(G));$$

$$f_a \geqq 0 \qquad (a\in\vec{A}\cup\overleftarrow{A}),$$

and its linear programming dual:

$$(3.6.12)\quad \max \sum_{u\in V(G)} w_u x_u$$

$$\text{s.t. } \pi_v - \pi_u + x_v \leqq 1 \qquad (\overrightarrow{uv}\in\vec{A});$$

$$\pi_u - \pi_v + x_u \leqq 0 \qquad (\overrightarrow{vu}\in\overleftarrow{A});$$

$$x_u \geqq 0 \qquad (v\in V(G)).$$

The theorem is proved with the help of the following three propositions:

<u>Proposition 1</u>: *The constraint matrix of (3.6.11) is totally unimodular. Consequently both (3.6.11) and (3.6.12) have integral optimal solutions (Hoffman and Kruskal [1956], cf. Theorem 1.2.15).*

<u>Proposition 2</u>: *Let* $\pi\in\mathbb{Z}^{V(G)}$, $x\in\mathbb{Z}^{V(G)}$ *be a feasible solution of (3.6.12). Then x is a feasible solution of the primal problem of (3.6.9).*

<u>Proposition 3</u>: *Let* $f\in\mathbb{Z}^{\vec{A}\cup\overleftarrow{A}}$ *be a feasible solution of (3.6.11). Then there exists a* $y\in\mathbb{Z}^{E(G)}$ *and a* $z\in\mathbb{Z}^{\Gamma(G)}$, *which form a feasible solution of the dual problem of (3.6.9), such that:*

$$\sum_{e\in E(G)} y_e + \sum_{C\in\Gamma(G)} \tfrac{1}{2}(|V(C)| -1)z_C \leqq \sum_{a\in\vec{A}} f_a .$$

Indeed, the three propositions together prove that $\alpha_w(G) \geqq \tilde{\rho}_w(G)$. By (3.6.7), this yields $\alpha_w(G) = \tilde{\rho}_w(G)$ and $\tau_w(G) = \tilde{\nu}_w(G)$. The three propositions above are shown as follows:

Proof of Proposition 1:

If we are given a directed graph $D = (V(D), A(D))$ and a spanning directed tree $T = (V(D), A(T))$ on the same node set (not necessarily $A(T) \subset A(D)$), then the *network matrix* N *of* D *with respect to* T is defined as follows: $N \in \{0,1,-1\}^{A(T) \times A(D)}$. For $u,v \in V(D)$ let $P(u,v) \subset A(T)$ be the unique path in T from u to v. Then for each $a_1 \in A(T)$, $a_2 = \overrightarrow{uv} \in A(D)$:

$$N_{a_1,a_2} := \begin{cases} 1 & \text{if } a_1 \in P(u,v)\text{, and } a_1 \text{ is passed forwardly going along } P(u,v) \\ & \text{from } u \text{ to } v; \\ -1 & \text{if } a_1 \in P(u,v)\text{, and } a_1 \text{ is passed backwardly going along} \\ & P(u,v) \text{ from } u \text{ to } v; \\ 0 & \text{if } a_1 \notin P(u,v). \end{cases}$$

Network matrices are totally unimodular (Tutte [1965]). We prove Proposition 1 by proving that the constraint matrix of (3.6.11) is a network matrix. Indeed, let $V(D) := V(T) := \{v_0\} \cup \{[u,i] \mid u \in V(G),\ i \in \{1,2\}\}$,
$A(D) := \{\overrightarrow{[u,1][v,2]} \mid \overrightarrow{uv} \in \vec{A}\}$, and
$A(T) := \{\overrightarrow{v_0[u,1]} \mid u \in V(G)\} \cup \{\overrightarrow{[u,1][u,2]} \mid u \in V(G)\}$.

Proof of Proposition 2:

Since x is integral we only need to prove that $x_u + x_v \leqq 1$ for $uv \in E(G)$. Indeed, $x_v + x_u \leqq (1 - \pi_v + \pi_u) + (\pi_v - \pi_u) = 1$ if $uv \in E(G)$ $(\overrightarrow{uv} \in \vec{A})$.

Proof of Proposition 3:

We can write f as $f = \sum_{D \in \Delta} \lambda_D f^D$, where Δ is a collection of directed circuits in $\vec{A} \cup \overleftarrow{A}$, $\lambda_D \in \mathbb{Z}_+$ for each $D \in \Delta$, and $f^D \in \{0,1\}^{\vec{A} \cup \overleftarrow{A}}$ with $f^D_a = 1$ if and only if $a \in D$.

For every even circuit $D\in\Delta$, let M_D be an arbitrary maximum cardinality matching in $\{uv\in E(G)\,|\,\overrightarrow{uv}\in D \text{ or } \overrightarrow{vu}\in D\}$. (In particular, if $D = \{\overrightarrow{uv},\overrightarrow{vu}\}$, then $M_D = \{uv\}$.) Define $y^D\in\mathbb{Z}^{E(G)}$ by:

$$y_e^D = \begin{cases}\lambda_D & \text{if } e\in M_D;\\ 0 & \text{else.}\end{cases}$$

Next $y\in\mathbb{Z}^{E(G)}$ is defined by:

$$y = \sum_{\substack{D\in\Delta\\ D \text{ even}}} y^D.$$

For each odd circuit $D\in\Delta$, let $C_D\in\Gamma(G)$ be defined by $C_D = \{uv\,|\,\overrightarrow{uv}\in D$ or $\overrightarrow{vu}\in D\}$. Define $z\in\mathbb{Z}^{\Gamma(G)}$ by:

$$z_C = \begin{cases}\lambda_D & \text{if } C = C_D \text{ for some } D,\ D\in\Delta,\ |D| \text{ odd;}\\ 0 & \text{else.}\end{cases}$$

The vectors $y\in\mathbb{Z}^{E(G)}$ and $z\in\mathbb{Z}^{\Gamma(G)}$ form a feasible solution to the dual problem of (3.6.9). Moreover

$$\begin{aligned}\sum_{a\in\vec{A}} f_a &= \sum_{D\in\Delta}\lambda_D|\vec{A}\cap D|\\ &\geqq \sum_{\substack{D\in\Delta\\ D\text{ even}}}\lambda_D|M_D| + \sum_{\substack{D\in\Delta\\ D\text{ odd}}}\lambda_D\cdot\tfrac{1}{2}(|V(C_D)|-1)\\ &= \sum_{e\in E(G)} y_e + \sum_{C\in\Gamma(G)}\tfrac{1}{2}(|V(C)|-1)z_C.\end{aligned}$$

□

<u>Proof of Theorem 3.6.8</u>:

Let G be a graph with no odd-K_4. Assume that all graphs G' with $|E(G')| < |E(G)|$ satisfy Theorem 3.6.8. We shall prove that G then satisfies Theorem 3.6.8. Obviously, we may assume G to be connected. Let $w\in\mathbb{Z}^{V(G)}$. By the weighted version of Theorem 3.6.5 we only need to prove that $\alpha_w(G) = \tilde{\rho}_w(G)$. Obviously we may assume that $w_u \geqq 0$ for each $u\in V(G)$.

According to Theorems 3.6.10 and 3.2.5 we may assume that G has a one-node cutset or a strong 2-split. So we have subsets V_1, V_2 of V(G) such that $|V_1\cap V_2| \leqq 2$, $V_1\cup V_2 = V(G)$, and both $V_1\setminus V_2$ and $V_2\setminus V_1$ are nonempty sets not joined by an edge in E(G). Moreover, in case $|V_1\cap V_2| = 2$, the subgraphs G_1 and G_2 in G induced by V_1, V_2 respectively are not bipartite. In the sequel we shall use the following notation: For each stable set $U \subset V_1 \cap V_2$ the number $s(U)$ ($s^1(U)$, $s^2(U)$ respectively) denotes the maximum weight $\sum_{u \in S} w_u$ of a stable set S in G (G_1, G_2 respectively) satisfying $S \cap V_1 \cap V_2 = U$. Note that: $s(U) = s^1(U) + s^2(U) - \sum_{u \in U} w_u$ for each stable set U in $V_1 \cap V_2$.

We consider two cases.

Case I: $V_1 \cap V_2$ *induces a complete subgraph in* G.

Define the following weight functions:

$$w_u^1 := \begin{cases} w_u & \text{if } u \in V_1 \setminus V_2; \\ w_u + s^1(\emptyset) - s^1(\{u\}) & \text{if } u \in V_1 \cap V_2; \end{cases}$$

$$w_u^2 := \begin{cases} w_u & \text{if } u \in V_2 \setminus V_1; \\ s^1(\{u\}) - s^1(\emptyset) & \text{if } u \in V_1 \cap V_2. \end{cases}$$

Obviously, neither G_1, nor G_2 contains an odd-K_4. Moreover $|E(G_1)| < |E(G)|$, $|E(G_2)| < |E(G)|$. Hence there exist a w^1- and a w^2-cover by edges and odd circuits in G_1, G_2 respectively, with cost $s^1(\emptyset)$, $\alpha_w(G) - s^1(\emptyset)$ respectively. The union of these two covers is a w-cover with edges and odd circuits in G with cost $\alpha_w(G)$. Hence $\alpha_w(G) = \tilde{\rho}_w(G)$.

Case II: $|V_1 \cap V_2| = 2$, $V_1 \cap V_2 = \{u_1, u_2\}$ *say, and* $u_1u_2 \notin E(G)$.

Define for i=1,2; k=2,3 the graph G_i^k by adding to G_i a path from u_1 to u_2 with k edges. (See Figures 3.14 and 3.15.)

Claim 1: *We may assume that* G_i^k *does not contain an* odd-K_4 (i=1,2; k=2,3). *Moreover*, $|E(G_i^k)| < |E(G)|$.

Proof of Claim 1: To prove the first assertion (for i=1), it is sufficient to prove that in G_2 there exists an odd as well as an even path from u_1 to u_2. Suppose this is not the case. Since G_2 is not bipartite this implies the existence of a cutnode in G_2 separating $\{u_1, u_2\}$ from an odd cycle in

G_2. But such a cutnode is also a cutnode of G. In that case we can apply Case I to prove $\alpha_w(G) = \tilde{\rho}_w(G)$. So we may assume that G_i^k has no odd-K_4.

If $|E(G_i^k)| \geq |E(G)|$, then $|E(G_2)| \leq 3$. Hence, since G_2 is not bipartite, G_2 is a triangle. So $u_1u_2 \in E(G)$, contradicting our assumption that $u_1u_2 \notin E(G)$.

<u>end of proof of claim 1</u>

Define $\Delta := s^2(\{u_1\}) + s^2(\{u_2\}) - s^2(\{u_1,u_2\}) - s^2(\emptyset)$. Again we consider two cases.

<u>Case IIa</u>: $\Delta \geq 0$.

Let b_1, b_2 be the new nodes in G_1^3, b the new node in G_2^2. (See Figure 3.14 below.) Moreover, let e_1, e_2, $\tilde{e}$, f_1, and f_2 be the edges indicated in Figure 3.14.

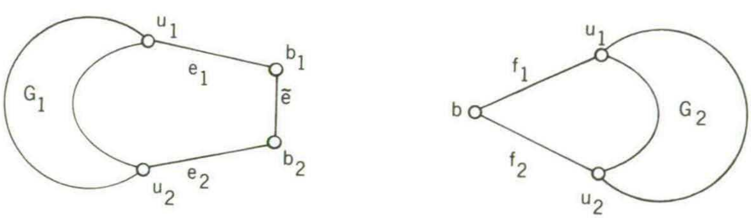


Figure 3.14

We define the following weight functions:

$$w^1 \in \mathbb{Z}^{V(G_1^3)} \text{ by } w_u^1 := \begin{cases} w_u & \text{if } u \in V_1 \setminus \{u_1,u_2\}; \\ s^2(\{u\}) - s^2(\emptyset) & \text{if } u \in \{u_1,u_2\}; \\ \Delta & \text{if } u \in \{b_1,b_2\}; \end{cases}$$

$$w^2 \in \mathbb{Z}^{V(G_2^2)} \text{ by } w_u^2 := \begin{cases} w_u & \text{if } u \in V_2 \setminus \{u_1,u_2\}; \\ w_u + s^2(\emptyset) - s^2(\{u\}) + \Delta & \text{if } u \in \{u_1,u_2\}; \\ \Delta & \text{if } u \in \{b\}. \end{cases}$$

<u>Claim 2</u>: $\alpha_{w^1}(G_1^3) = \alpha_w(G) + \Delta - s^2(\emptyset)$ *and* $\alpha_{w^2}(G_2^2) = s^2(\emptyset) + \Delta$.

Moreover, for i=1,2 *there exists a stable set* S *in* G_2^2 *with* $\sum_{u\in S} w_u^2 = \alpha_{w^2}(G_2^2)$, $u_i \notin S$, *and* $b \notin S$.

Proof of Claim 2: Straightforward casechecking.

end of proof of claim 2

By Claim 1 there exists a w^1-cover E^1, Γ^1 by edges and odd circuits in G_1^3 with cost $\alpha_{w^1}(G_1^3) = \alpha_w(G) + \Delta - s^2(\emptyset)$. Let γ_1, γ_2 and $\tilde{\gamma}$ denote the multiplicity of e_1, e_2, $\tilde{e}$ respectively in E^1. Let β denote the sum of the multiplicities of the odd cycles in Γ^1 containing b_1 (and b_2). Assume E^1 and Γ^1 are such that $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$ is minimal.

Claim 3: $\gamma_i + \tilde{\gamma} + \beta = \Delta$ *for* i=1,2. *Consequently*, $\gamma_1 = \gamma_2$.

Proof of Claim 3: $\gamma_i + \tilde{\gamma} + \beta \geqq \Delta$, since E^1, Γ^1 is a w^1-cover. Suppose $\gamma_1 + \tilde{\gamma} + \beta > \Delta$. Then $\tilde{\gamma} = 0$. Indeed, if not, then increasing γ_2 by 1 and decreasing $\tilde{\gamma}$ by 1 would yield a w^1-cover with cost $\alpha_{w^1}(G_1^3)$, and smaller $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$. Moreover, $\gamma_1 = 0$. Otherwise, take some $u_1v\in E(G^1)$. Adding u_1v to E^1 (or increasing its multiplicity in E^1) and decreasing γ_1 by 1, again yields a w^1-cover with cost $\alpha_{w^1}(G_1^3)$, and smaller $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$. Finally, $\beta = 0$, contradicting the fact that $\Delta \geqq 0$. Indeed, if $\beta > 0$ remove an odd circuit C with $b_1\in V(C)$ from Γ^1, and add the edges in the unique maximum cardinality matching $M\subseteq E(C)$ not covering b_1, to E^1. Since $M = \frac{1}{2}(|V(C)| - 1)$ this again yields a w^1-cover with cost $\alpha_{w^1}(G_1^3)$, and smaller $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$.

end of proof of claim 3

By Claim 1, there also exists a w^2-cover E^2, Γ^2 by edges and odd circuits in G_2^2 with cost $\alpha_{w^2}(G_2^2) = s^2(\emptyset) + \Delta$. Let E^2 and Γ^2 be such that the sum, δ say, of the multiplicities of the odd cycles in Γ^2 containing b is minimal.

Claim 4: f_1 *and* f_2 *do not occur (i.e. have multiplicity* 0) *in* E^2. *Moreover*, $\delta = \Delta$.

Proof of Claim 4: Since the cost of E^2, Γ^2 is $\alpha_{w^2}(G_2^2)$ and there exists a stable set S in G_2^2 with $\sum_{u\in S} w_u^2 = \alpha_{w^2}(G_2^2)$ and $u_1, b\notin S$ (Claim 2), the edge f_1

does not occur in E^2 ("complementary slackness"). Equivalently f_2 does not occur in E^2. The proof that $\delta = \Delta$ is similar to the proof of Claim 3.

<u>end of proof of claim 4</u>

Using E^1, Γ^1 and E^2, Γ^2 we are now able to construct a w-cover $\tilde{E}$, $\tilde{\Gamma}$ in G by edges and odd circuits, and with cost $\alpha_w(G)$, thus proving $\alpha_w(G) = \tilde{\rho}_w(G)$. The construction goes as follows:

<u>Step 1</u>: The edges in E^1 and E^2, except e_1, e_2 and $\tilde{e}$, are added to $\tilde{E}$ (with the same multiplicity). The odd circuits in Γ^1 and Γ^2 not containing b_1 (b_2), or b are added to $\tilde{\Gamma}$.

<u>Step 2</u>: Let $C_1^2,\ldots,C_\Delta^2$ be the odd circuits in Γ^2 containing b. (Remember that some of them may be equal.)

(i) Let $C_i^1,\ldots,C_\beta^1$ be the odd circuits in Γ^1 containing b_1. Define for each $i=1,\ldots,\beta$ the odd circuit $C_i \in \Gamma(G)$ by $E(C_i) = E(C_i^1) \cup E(C_i^2) \setminus \{e_1, e_2, \tilde{e}, f_1, f_2\}$. Add all the odd circuits $C_1,\ldots,C_\beta$ to $\tilde{\Gamma}$.
Note that, for each $i=1,\ldots,\beta$: $\frac{1}{2}(|V(C_i)| - 1) = \frac{1}{2}(|V(C_i^1)| - 1) + \frac{1}{2}(|V(C_i^2)| - 1) - 2$.

(ii) Define for each $i=\beta+1,\ldots,\beta+\gamma_1$ the collection of edges M_i as the unique maximum cardinality matching in $E(C_i^2)$ not covering b. Each edge occuring in M_i ($i=\beta+1,\ldots,\beta+\gamma_1$) is added to $\tilde{E}$ (as often as it occurs in any M_i).
Note that, for each $i=\beta+1,\ldots,\beta+\gamma_1$: $|M_i| = \frac{1}{2}(|V(C_i^2)| - 1)$.

(iii) Define for each $i=\beta+\gamma_1+1,\ldots,\beta+\gamma_1+\tilde{\gamma}$ ($= \Delta$) the collection of edges N_i as the unique maximum cardinality matching in $E(C_1^2)$ not covering u_1 and not covering u_2. All the edges occuring in any N_i are added to $\tilde{E}$ (as often as they occur in any N_i).
Note that, for each $i=\beta+\gamma_1+1,\ldots,\Delta$, $|N_i| = \frac{1}{2}(|V(C_1^2)| - 1) - 1$.

<u>Claim 5</u>: *The collections* $\tilde{E}$, $\tilde{\Gamma}$ *form a* w*-cover by edges and odd circuits in* G.

<u>Proof of Claim 5</u>: It is not hard to see that each $u \in (V_1 \setminus V_2) \cup (V_2 \setminus V_1)$ is covered w_u times by $\tilde{E}$, $\tilde{\Gamma}$. (The matchings in step 2(ii) and in step 2(iii) of the construction do not decrease the number of times that a node in

$V_2 \setminus V_1$ is covered.) The node u_1 is covered as least $s^2(\{u\}) - s^2(\emptyset)$ times by E^2, Γ^2, and at least $w_u + s^2(\emptyset) - s^2(\{u\}) + \Delta$ times by E^1, Γ^1. So u_1 is covered at least $w_u + \Delta$ times by E^1, Γ^1 and E^2, Γ^2 together. During the construction this amount is decreased with β by step 2(i), with γ_1 by step 2(ii), and with $\tilde{\gamma}$ by step 2(iii). Since $\beta + \gamma_1 + \tilde{\gamma} = \Delta$, $\tilde{E}$ and $\tilde{\Gamma}$ cover u_1 at least w_u times. Similarly one deals with u_2, as $\gamma_1 = \gamma_2$.

end of proof of claim 5

Claim 6: *The cost of* $\tilde{E}$, $\tilde{\Gamma}$ *is* $\alpha_w(G)$.

Proof of Claim 6: The cost of E^1, Γ^1 plus the cost of E^2, Γ^2 is equal to $\alpha_{w^1}(G_1^3) + \alpha_{w^2}(G_2^2) = \alpha_w(G) + \Delta - s^2(\emptyset) + \Delta = \alpha_w(G) + 2\Delta$. During the construction we lost exactly: 2β in step 2(i), $\tilde{\gamma}$ in step 2(iii), and $2\gamma_1 + \tilde{\gamma}$ by ignoring the edges e_1, e_2, $\tilde{e}$. so the cost of $\tilde{E}$, $\tilde{\Gamma}$ is $\alpha_w(G) + 2\Delta - 2\beta - \tilde{\gamma} - (2\gamma_1 + \tilde{\gamma}) = \alpha_w(G)$.

end of proof of claim 6

Claims 5 and 6 together yield that $\alpha_w(G) = \tilde{\rho}_w(G)$.

Case IIb: $\Delta \leqq 0$.

The proof of this case is similar to the proof of Case IIa. Therefore we shall only give the beginning of it.

Let b the new node in G_1^2 and let b_1 and b_2 be the new nodes in G_2^3 (see Figure 3.15).

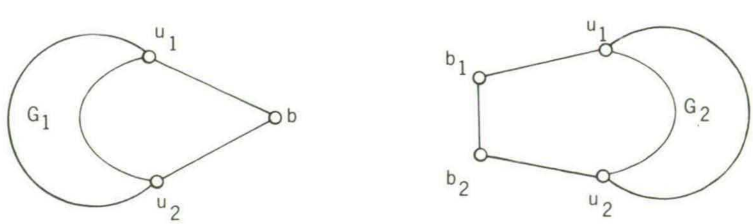


Figure 3.15

Define the following weight functions:

$$w^1 \in \mathbb{Z}^{V(G_1^2)} \text{ by } w_u^1 := \begin{cases} w_u & \text{if } u \in V_1 \setminus V_2; \\ s^2(\{u\}) - s^2(\emptyset) - \Delta & \text{if } u \in \{u_1, u_2\}; \\ -\Delta & \text{if } u = b; \end{cases}$$

$$w^2 \in \mathbb{Z}^{V(G_2^3)} \text{ by } w_u^2 := \begin{cases} w_u & \text{if } u \in V_2 \setminus V_1; \\ w_u + s^2(\emptyset) - s^2(\{u\}) & \text{if } u \in \{u_1, u_2\}; \\ -\Delta & \text{if } u \in \{b_1, b_2\}. \end{cases}$$

The first thing to be proved now is

<u>Claim 7</u>: $\alpha_{w^1}(G_1^2) = \alpha_w(G) - \Delta - s^2(\emptyset)$ *and* $\alpha_{w^2}(G_2^3) = -\Delta + s^2(\emptyset)$. *Moreover, for each* $U \in \{\{u_1,b_1\}, \{b_1,b_2\}, \{u_2,b_2\}\}$ *there exists a stable set* S *in* G_2^3 *with* $\sum_{u \in S} w_u = \alpha_{w^2}(G_2^2)$, *and* $S \cap U = \emptyset$.

From this point it is not hard to see how arguments similar to those used in Case IIa prove that $\alpha_w(G) = \tilde{\rho}_w(G)$. □

<u>Remarks on the proof of Theorem 3.6.8</u>:

The proof of Case I of the proof above is identical with the proof of Theorem 4.1 in Chvátal [1975]. The techniques used in case IIa and Case IIb of the proof are similar to the techniques used by Boulala and Uhry [1979]. However, they restrict G_2 to paths and odd cycles. Sbihi and Uhry [1984] also use the decompositions of Case II. However, they used these decompositions in case G_2 bipartite. Recently, Barahona and Mahjoub [1986] derived a construction to derive all facets of the stable set polytope of G, in case G has a two node cutset $\{u_1,u_2\}$, from the facets of the stable set polytopes of G_1^+, and G_2^+. (Here G_1 and G_2 are as in the proof above, G_i^+ is derived from G_i by adding a five cycle $\{u_1,b,u_2,b_1,b_2\}$).

Next we give some remarks on the min-max relations in Theorem 3.6.3 and 3.6.8.

Remarks:

(i) Theorem (3.6.8) implies that if G contains no odd-K_4, then $\tilde{\rho}_w(G) = \tilde{\rho}^*_w(G)$ for each $w \in \mathbb{Z}_+^{V(G)}$. In other words, the system of linear inequalities in the primal problem of (3.6.9) is totally dual integral. Consequently, if G contains no odd-K_4, then $\alpha_w(G) = \tilde{\rho}^*_w(G)$ for each $w \in \mathbb{Z}_+^{V(G)}$. This means that the system of linear inequalities in the primal problem of (3.6.9) describes the stable set polytope of G. Obviously, also the statement "$\tau_w(G) = \tilde{\nu}_w(G)$ for each $w \in \mathbb{Z}_+^{V(G)}$" can be formulated in a way similar to (3.6.9).

(ii) Theorem 3.6.8 (and Theorem 3.6.3) can be refined by allowing w-covers (w-packings) by edges and odd circuits only to use edges not contained in a triangle, and odd circuits not having a chord. In other words, if G has no odd-K_4, then the system:

$$
\begin{array}{lll}
 & x_u + x_v \leqq 1 & (uv \in E(G),\ uv \text{ is not contained in a triangle}); \\
(*) & \sum\limits_{u \in V(C)} x_u \leqq \tfrac{1}{2}(|V(C)| - 1) & (C \in \Gamma(G),\ C \text{ has no chord}); \\
 & x_u \geqq 0 & (u \in V(G)),
\end{array}
$$

is a totally dual integral system defining the stable set polytope of G. In fact the inequalitities in (*) are all facets of the polyhedron defined by (*) (for any graph G). So (*) is the unique minimal totally dual integral system (cf. Schrijver [1981] (,see Theorem 1.2.21 (ii) of this monograph)) for the stable set polytope of G, in case G has no odd-K_4.

(iii) Earlier results on this topic are:

- Chvátal [1975]: If G is *series-parallel* (i.e. G contains no homeomorph of K_4), then $\alpha(G) = \tilde{\rho}(G)$.
- Boulala and Uhry [1979]: If G is series-parallel, then $\alpha_w(G) = \tilde{\rho}_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$. (In fact they only emphasize $\alpha_w(G) = \tilde{\rho}^*_w(G)$ (which was conjectured by Chvátal [1975]), but their proof implicitly yields the stronger result. Recently, Mahjoub [1988] gave a

very short proof of $\alpha_w(G) = \widetilde{\rho}^*_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$ for series-parallel graphs G.)

- Fonlupt and Uhry [1982]: G is almost bipartite, then $\alpha_w(G) = \widetilde{\rho}^*_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$. Sbihi and Uhry [1984] give a new proof of Fonlupt and Uhry's result. This proof implicitly yields $\alpha_w(G) = \widetilde{\rho}_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$.

Obviously, the graphs considered by Chvátal, Boulala, Fonlupt, Sbihi, and Uhry do not contain an odd-K_4.

- Gerards and Schrijver [1985]: If G has no odd-K_4 then $\alpha_w(G) = \widetilde{\rho}^*_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$ (cf. Theorem 2.3.3).

The last remark states that Theorem 3.6.8 implies that the polyhedron defined by

$$\begin{cases} x_u \geqq 0 & u \in V(G); \\ x_u + x_v \leqq 1 & uv \in E(G), \end{cases}$$

has Chvátal rank 1 in case G has no odd-K_4. In fact, Theorem 3.6.8 yields a new proof of Theorem 2.3.3.

Let A is a bidirected graph with no odd-K_4 and let

$$P := \{x \mid a \leqq x \leqq b,\ c \leqq Ax \leqq d\}$$

with a,b,c and d integral vectors. Then it is easy to see that P' is the projection of a face of Q' where

$$Q = \{x \mid x_u \geqq \widetilde{a}_u,\ u \in V(G),\ x_u + x_v \leqq \widetilde{b}_{uv},\ uv \in E(G)\},$$

and G a suitable graph.

[Indeed, by replacing the inequalities in $a \leqq x \leqq b$, $c \leqq Ax \leqq d$, by new inequalities, hereby introducing new variables (if necessary). This replacement is as follows:

$$
\begin{aligned}
x_i + x_j \leqq \gamma &\longrightarrow x_i + x_j \leqq \gamma;\\
- x_i + x_j \leqq \gamma &\longrightarrow x_i + x_{ij}^{i} = 0,\ x_{ij}^{i} + x_j \leqq \gamma;\\
- x_i - x_j \leqq \gamma &\longrightarrow x_i + x_{ij}^{i} = 0,\ x_{ij}^{i} + x_{ij}^{j} \leqq \gamma,\ x_{ij}^{j} + x_j = 0;\\
x_i \geqq \alpha &\longrightarrow x_i \geqq \alpha;\\
x_i \leqq \beta &\longrightarrow x_i + y_i = 0,\ y_i \geqq - \beta.
\end{aligned}
$$
]

It is obvious, from the indicated construction that P' is an integral polyhedron if Q' is, and that the constructed undirected graph G contains no odd-K_4. To prove that Q' is integral, let z be a vertex of Q'. Obviously we may assume that $0 \leqq z_u < 1$ ($u \in V(G)$) (by translating Q). Moreover, Q' is constrained by the inequalities:

$$
\begin{aligned}
&x_u \geqq \tilde{a}_u && (u \in V(G));\\
&x_u + x_v \leqq b_{uv} && (e \in E(G));\\
&\sum_{u \in V(C)} x_u \leqq \tfrac{1}{2}\Big[\sum_{a \in E(C)} b_e - 1\Big] && (C \in \Gamma(G)).
\end{aligned}
$$

We may assume that $\tilde{a}_u = 0$ for $u \in V(G)$, and (like in the proof of Theorem 2.3.3) that $b_e \in \{0,1\}$ for $e \in E(G)$. Hence, by Theorem 3.6.8, Q' is the stable set polytope of G. So z is an integral vector, which completes the proof of Theorem 2.3.3.

COMPUTATIONAL ASPECTS

We conclude this section by paying some attention to the computational complexity of the problems: Given G and $w \in \mathbb{Z}^{V(G)}$, determine $\alpha_w(G)$, $\tilde{\rho}_w(G)$, $\rho_w(G)$, $\tau_w(G)$, $\tilde{\nu}_w(G)$, and $\nu_w(G)$. Well-known results are:

- It is $\mathcal{NP}$-hard to determine $\alpha_w(G)$, $\tau_w(G)$, even if $w \equiv 1$ (Karp [1972]).
- There exists a polynomial-time algorithm to determine a maximum cardinality w-matching, or a minimum cardinality w-edge-cover (Edmonds [1965a] for $w \equiv 1$, Cunningham and Marsh [1978] for general w).

Pulleyblank [personal communication] observed that determining $\tilde{\rho}_w(G)$, or $\tilde{\nu}_w(G)$ is $\mathcal{NP}$-hard, even if $w \equiv 1$. There is a reduction from PARTITION INTO TRIANGLES (cf. Garey and Johnson [1979]). Indeed, given a graph G there is partition of V(G) into triangles in G if and only if $\tilde{\rho}(G) \leqq \frac{1}{3}|V(G)|$. Since PARTITION INTO TRIANGLES remains $\mathcal{NP}$-complete for planar graphs (Dyer and

Frieze [1986]), determining $\tilde{\rho}(G)$, or $\tilde{\nu}(G)$, remains $\mathcal{NP}$-hard even if G is planar.

If G contains no odd-K_4, then $\tilde{\rho}_w(G)$ and $\tilde{\nu}_w(G)$ can be found in polynomial-time. Indeed, an algorithm can be obtained from the proofs given above (proofs of Theorem 3.6.10 and 3.6.8). However there are some difficulties to be settled.

SOLVING (3.6.11) AND (3.6.12)

If G has an orientation of discrepancy 1, such orientation can be found in polynomial-time (see the final remarks in Section 3.3). Having this orientation $\vec{A}$ one can solve (3.6.11) and (3.6.12) as follows: Define the directed graph $D = (V(D), A(D))$ by: $V(D) := \{u_i | u \in V(G);\ i=1,2\}$, and $A(D) := A_1(D) \cup A_2(D)$, with $A_1(D) := \{\overrightarrow{u_1u_2} | u \in V(G)\}$ and $A_2(D) := \{\overrightarrow{u_2v_1} | u,v \in V(G), \overrightarrow{uv} \in \vec{A}\}$. Then (3.6.11) is equivalent to the min-cost-circulation problem:

$$(3.6.13)\quad \min \sum_{a \in A_2(D)} g_a$$
$$\text{s.t. } g \text{ is a nonnegative circulation in } D,$$
$$\text{and } g_{\overrightarrow{u_1u_2}} \geqq w_u \quad (u \in V(D)).$$

(3.6.13) can be efficiently solved by the out-of-kilter method of Ford and Fulkerson [1962]. (Note that since the costfunction is {0,1}-valued, there is no need to appeal to more sophisticated techniques as used by Edmonds and Karp [1972], Röck [1980] or Tardos [1985].)

DECOMPOSITION

If G has no orientation of discrepancy 1, then it has a one or two node cutset (with, in the latter case, both sides not bipartite). We can now go along the lines of Cases I and II in the proof of Theorem 3.6.8. In this way we get a recursive algorithm. However, in one side of the decomposition we have to solve two or three stable set problems to determine the numbers s'(U). (See the proof of Theorem 3.6.8.) Next we have to solve a stable set problem on both parts of the decomposition. If solving all of these four or five problems again needs a decomposition this might lead to an exponential number of steps. However there is a way to avoid this. Any time we have to decompose the graph we search a decomposition in which the smallest side, G_1 say, is as small as possible. In that case G_1^2 and G_1^3 have an orientation of discrepancy 1. So the two or three stable set problems to determine the numbers s'(U) as well as the derived problems on G_1^2 or G_1^3 can be solved without further recursion. If we organize our algorithm in this way there is no risk for exponential explosion.

CHAPTER 4. T-JOINS

In this chapter we consider T-joins. Beside an introductory chapter, giving definitions and a short survey of the literature, this chapter consists of two parts. In Section 4.2 we give a common generalization of two theorems of Seymour on T-joins. Here, again, the odd-K_4's play a role. In Sections 4.3 until 4.5 we study the properties of a binary matroid associated with T-joins in a graph. In parallel with Sections 3.1 until 3.3 we give decomposition results and orientation results for specially structured T-join problems. The results in Section 4.5 are applied in Section 4.6 to give new proofs of certain min-max relations for specially structured T-join problems.

4.1. INTRODUCTION TO T-JOINS IN GRAPHS

Let G be an undirected graph, and let T be a subset of V(G). A *T-join* is a subset F of E(G), such that $\{v \in V(G) \,\big|\, |\delta(v) \cap F| \text{ is odd}\} = T$. Obviously, if G is connected then there exists a T-join if and only if $|T|$ is even. More generally there exists a T-join if and only if $|T \cap V(G_1)|$ is even for each component G_1 of G. If $U \subset V$, such that $|U \cap T|$ is odd, then $\delta(U)$ is called a T-*cut*. We define:

$\nu_T(G)$: = maximum cardinality of a collection of disjoint T-cuts;

$\tau_T(G)$: = minimum cardinality of a T-join.

Obviously $\nu_T(G) \leqq \tau_T(G)$, since for each T-join F and each T-cut $\delta(U)$, we have $|F \cap \delta(U)| \geqq 1$.

<u>Theorem 4.1.1</u> (Seymour [1981])
Let G be a connected bipartite graph. Then for each even $T \subset V(G)$: $\nu_T(G) = \tau_T(G)$. □

We omit the proof of Theorem 4.1.1. Proofs can be found in Seymour [1981], Frank, Sebö, and Tardos [1984], and Sebö [1985b]. Theorem 4.1.1 yields the following min-max relation for T-joins in general graphs.

<u>Theorem 4.1.2</u> (Edmonds and Johnson [1970, 1973], Lovász [1975])
Let G be an undirected connected graph. If T is an even subset of V(G), then $2\tau_T(G)$ is equal to the maximum number of T-cuts such that each edge occurs in at most two of them.

<u>Proof</u>: Apply Theorem 4.1.1 to the bipartite graph G' and $T' \subset V(G')$ defined as follows

$V(G') := V(G) \cup E(G)$;
$E(G') := \{ue \,|\, u \in V(G), e \in E(G),\ u \text{ endpoint of } e\}$;
$T' := T$. □

There are two, at first sight unrelated, special cases of T-joins.

THE CHINESE POSTMAN PROBLEM

Given a graph G, a *chinese postman tour* is a sequence of nodes $v_0, v_1, \ldots, v_k = v_0$ such that $v_{i-1}v_i \in E(G)$ $(i=1,\ldots,k)$, and for each $e \in E(G)$ there exists an $i=1,\ldots,k$ such that $e=v_{i-1}v_i$. It is not hard to see that the minimum length of a chinese postman tour with respect to some given length function $w \in \mathbb{Z}^{E(G)}$ is equal to $\sum_{e \in E(G)} w_e + \min \{ \sum_{e \in F} w_e \mid F \text{ is T-join}\}$, where $T := \{u \in V(G) \mid |\delta_G(u)| \text{ is odd}\}$. Edmonds and Johnson [1973] derived Theorem 4.1.2 in the context of the chinese postman problem. (It is easy to see that this is not really a restriction.)

MULTICOMMODITY FLOWS IN PLANAR GRAPHS

Lemma 4.1.3 (Guan [1962])
Let G *be graph and* $T \subset V(G)$ *with* $|T|$ *even. Then a* T*-join* F *in* G *is a minimum cardinality* T*-join, if and only if* $|E(C) \cap F| \leqq |E(C) \setminus F|$ *for each circuit* C *in* G.

Proof: This lemma is an easy consequence of the following observation: if F_1 and F_2 are T-joins in G then $F_1 \Delta F_2$ is a cycle in G. □

The following observation is easy to prove too.

Lemma 4.1.4
Let G *be a graph, and* $T \subset V(G)$ *with* $|T|$ *even. Let* F *be a minimum cardinality* T*-join. Then* $\tau_T(G) = \nu_T(G)$ *if and only if there exists a collection edge disjoint coboundaries* $\delta(U_f)$ $(f \in F)$ *such that* $f \in \delta(U_f)$ *for each* $f \in F$.

□

These two simple observations will turn out to be useful. First for understanding the relation between T-joins and multicommodity flows in planar graphs, and later in the proof of Theorem 4.2.2, which is an extension of Theorem 4.1.1.

Let G be a graph, and $D \subseteq E(G)$. The *multicommodity flow problem* in G with respect to D is: does there exist a collection of edge disjoint circuits C_d ($d \in D$) in G such that $d \in C_d$ ($d \in D$)? A necessary condition obviously is the *cut-condition:* $|\delta(U) \cap D| \leqq |\delta(U) \setminus D|$ for each $U \subseteq V(G)$. However the condition need not be sufficient, as is shown in Figure 4.1 (with $D = \{d_1, d_2\}$).

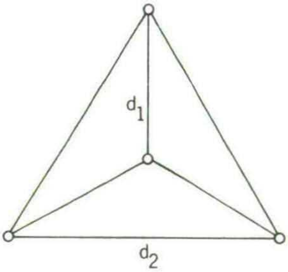


Figure 4.1

Let us suppose now that G is planar. Let G^* be a planar dual of G (with respect to some embedding of G in the plane). We may identify E(G) and $E(G^*)$, and consider $D \subseteq E(G^*)$. Now Lemma 4.1.3 shows that D satisfies the cut-condition in G if and only if D is a minimal T(D)-join in G^* (where T(D) is the collection of those nodes in G^* that are endpoints of an odd number of edges in D). Moreover, the existence of the desired circuits in the multicommodity flow problem in G with respect to D is equivalent to $\nu_{T(D)}(G^*) = |D|$. So we get: If D satisfies the cut-condition in G then there exists a collection of edge disjoint circuits C_d($d \in D$) with $d \in C_d$ for each $d \in D$, if and only if $\nu_{T(D)}(G^*) = \tau_{T(D)}(G^*)$. In particular, with Theorem 4.1.1, this implies: (*Eulerian* graph = connected cycle.)

<u>Theorem 4.1.5</u> (Seymour [1981])
Let G be an eulerian planar graph, and let $D \subseteq E(G)$. Then there exists a collection of edge disjoint circuits C_d($d \in D$) such that $d \in C_d$ for each $d \in D$ if and only if D satisfies the cut-condition in G.

<u>Proof</u>: If G is eulerian, G^* is bipartite. So the theorem follows from Theorem 4.1.1 and the discussion above. □

This relation between T-cuts and multicommodity flows forms a motivation for the study of those graphs G for which $\nu_T(G) = \tau_T(G)$ for all even $T\subseteq V(G)$. This is the subject of Section 4.2.

We close this section with a description of Edmonds' algorithm to find a minimum weight T-join.

Let G be an undirected graph, $T\subseteq V(G)$ with $|T|$ even, and $\ell\in \mathbb{Z}_+^{E(G)}$. The following algorithm finds a T-join F which minimizes $\sum_{e\in F} \ell_e$.

EDMONDS SHORTEST T-JOIN ALGORITHM (Edmonds [1965d], cf. Edmonds and Johnson [1973]).
Let H be the simple complete graph with V(H) = T. For each $s,t\in T$ find a shortest st-path, P_{st}, in G with respect to ℓ. Let $w_{st} := \sum_{e\in P_{st}} \ell_e$ for each $s,t\in T$. Find a minimum weight perfect matching $s_1t_1, s_2t_2,\ldots,s_kt_k$ in H (with respect to w, where $k = \frac{1}{2}|T|$). Let $F := E(P_{s_1t_1})\Delta\ldots\Delta E(P_{s_kt_k})$. Then F is a shortest T-join.

If one uses polynomial-time algorithms to find the shortest path P_{st} and the minimum weight perfect matching $s_1t_1,\ldots,s_kt_k$, then the shortest T-join algorithm above is polynomial-time. (Polynomial-time shortest path algorithms are Dijkstra's algorithm (Dijkstra [1959]) and the Floyd-Warshall algorithm (Floyd [1962] and Warshall [1962]). Edmonds algorithm for minimum weight perfect matching is polynomial-time (Edmonds [1965c]).)

Remarks:

Sebö [1985a, 1986] describes a good characterization for shortest paths in a weighted undirected graph with no negatively weighted circuits (edges may have negative weight). Using this, Sebö proves a structure theorem for T-joins, generalizing the Edmonds-Gallai structure of matchings (Edmonds [1965a], Gallai [1963, 1964]).

4.2. A COMMON GENERALISATION OF TWO THEOREMS OF SEYMOUR ON T-JOINS

In this section we study graphs G for which $\nu_T(G) = \tau_T(G)$ for each even $T \subset V(G)$. From Theorem 4.1.1 and Theorem 4.6.1 we have

<u>Theorem 4.2.1</u> (Seymour [1981, 1977])
Let G *be a connected graph. If* G *is bipartite or series-parallel, then* $\nu_T(G) = \tau_T(G)$ *for each even subset* T *of* V(G). $\square$

[If G is series-parallel, then for each $T \subset V(G)$ the graft [G,T] (cf. Section 4.3) has no K_4-partition (cf. Section 4.3). So, by Theorem 4.6.1, $\nu_T(G) = \tau_T(G)$ for each even subset $T \subset V(G)$.]

Theorem 4.2.1 provides two sufficient conditions for $\nu_T(G) = \tau_T(G)$. These two conditions are of a quite different nature: bipartiteness is a parity condition (all circuits are even), whereas being series-parallel is a topological condition (no homeomorph of K_4 as a subgraph). The following theorem replaces these two sufficient conditions by one weaker condition:

<u>Theorem 4.2.2</u>
Let G *be an undirected, connected graph. If* (G,E(G)) *contains neither an odd-*K_4 *nor an odd-prism, then for each even* $T \subset V(G)$ *we have* $\nu_T(G) = \tau_T(G)$. $\square$

(We prove this result later in this section.)

Here an *odd-prism* is a (signed) graph as depicted in Figure 4.2. Wriggled lines stand for pairwise openly disjoint paths, while **odd**, **even** respectively indicates that the corresponding faces are odd circuits, even circuits respectively.

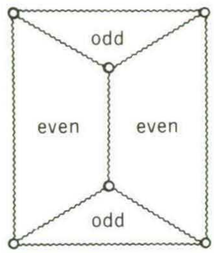


Figure 4.2

It is straightforward to see that neither bipartite graphs, nor series-parallel graphs contain an odd-K_4 or an odd-prism. So Theorem 4.2.2 implies Theorem 4.2.1. The two forbidden configurations odd-K_4 and odd-prism are motivated by the fact that $\nu_{V(G)}(G) \neq \tau_{V(G)}(G)$ in case $G = K_4$ or G is the triangular prism (Figure 4.3).

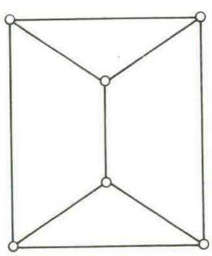


Figure 4.3

Remark:

The condition in Theorem 4.2.2 is not a necessary condition since $\nu_T(G) = \tau_T(G)$ for all $T \subset V(G)$ for the odd-K_4 in Figure 4.4.

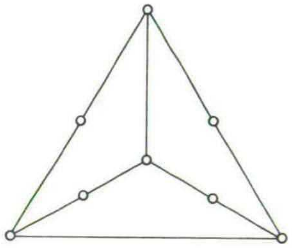


Figure 4.4

However from Theorem 4.2.2 one can derive:
Let (G,Σ) be a signed graph. Then the following are equivalent:

(i) *(G,Σ) contains no odd-K_4 and no odd-prism;*
(ii) *For each weight function $w \in \mathbb{Z}_+^{E(G)}$ with the property that $\sum_{e \in E(G)} w_e$ is even if and only if C is an even circuit in (G,Σ), we have: for each even $T \subset V(G)$ the minimum weight of a T-join with respect to w is equal to the maximum cardinality of a w-packing of T-cuts.*

To prove Theorem 4.2.2 we use the following theorem.

Theorem 4.2.3
Let (G,Σ) be a signed graph with no odd-K_4 and no odd-prism. If G is simple then one of the following holds:
(i) (G,Σ) has a 1-split;
(ii) (G,Σ) has a strong 2-split;
(iii) (G,Σ) is almost bipartite.

Proof: Let (G,Σ) satisfy the conditions of the theorem, without satisfying (i) or (ii). We prove that G is almost bipartite.

Claim 1: *There are no two node disjoint odd circuits.*
Proof of Claim 1: Suppose to the contrary that C_1 and C_2 are odd circuits with $V(C_1) \cap V(C_2) = \emptyset$. Obviously $|V(C_i)| \geq 3$ for i = 1,2 (as G is simple). Since (i) and (ii) are not satisfied, Menger's Theorem (Menger [1927], cf.

Theorem 1.3.1) yields the existence of three paths P_1, P_2 and P_3 from C_1 to C_2 such that $V(P_i) \cap V(P_j) = \emptyset$ $(i,j = 1,2,3,\ i \neq j)$. It is easy to see that C_1, C_2, P_1, P_2 and P_3 together form an odd-prism or contain an odd-K_4. This is a contradiction. <u>end of proof of claim 1</u>

For each odd circuit C in (G,Σ) and each bridge $B \subset E(G)$ of C there exists a unique path $I_C(B)$ on C with the following properties:

- there exists an odd circuit C' such that $E(C') \subset E(C) \cup B$; $V(C) \cap V(C') = V(I_C(B))$ and $E(C) \cap E(C') = E(I_C(B))$;
- each odd circuit C' with $E(C') \subset E(C) \cup B$ satisfies: $V(C) \cap V(C') \supset V(I_C(B))$ and $E(C) \cap E(C') \supset E(I_C(B))$.

Indeed if C contains at least three nodes with degree at least three, this follows from Claim 1 and Lemma 1.3.5. If C contains at most two nodes of degree at least three, this follows from the fact that (G,Σ) has no 1-split and no strong 2-split. Note that it might be the case that $|V(I_C(B))| = 1$ and $E(I_C(B)) = \emptyset$.

Now choose an odd circuit $\tilde{C}$ and a bridge $\tilde{B}$ of $\tilde{C}$, such that $I_{\tilde{C}}(\tilde{B})$ has a minimal number of edges, among all $I_C(B)$ (over all odd circuits C, and bridges B of C). Let $\tilde{u}$ be an endpoint of $I_{\tilde{C}}(\tilde{B})$.

<u>Claim 2</u>: $\tilde{u} \in V(I_{\tilde{C}}(B))$ *for each bridge* B *of* $\tilde{C}$.

<u>Proof of Claim 2</u>: Suppose to the contrary that $\tilde{u} \notin V(I_{\tilde{C}}(B))$ for some bridge B of $\tilde{C}$. Since $I_{\tilde{C}}(\tilde{B})$ is minimal, $V(I_{\tilde{C}}(B)) \setminus V(I_{\tilde{C}}(\tilde{B})) \neq \emptyset$. Let $u \in V(I_{\tilde{C}}(B)) \setminus V(I_{\tilde{C}}(\tilde{B}))$.

Let $\hat{C}$ be an odd circuit, with $E(\hat{C}) \subset E(\tilde{C}) \cup \tilde{B}$, $V(\hat{C}) \cap V(\tilde{C}) = V(I_{\tilde{C}}(\tilde{B}))$, and $E(\hat{C}) \cap E(\tilde{C}) = E(I_{\tilde{C}}(\tilde{B}))$. Similarly, let C be an odd circuit, with $E(C) \subset E(\tilde{C}) \cup B$, $V(C) \cap V(\tilde{C}) = V(I_{\tilde{C}}(B))$, and $E(C) \cap E(\tilde{C}) = E(I_{\tilde{C}}(B))$. Obviously $u \notin V(\hat{C})$. Let $\hat{B}$ be the bridge of $\hat{C}$ containing u. Then $E(C)$ is contained in $\hat{B} \cup E(\hat{C})$. So $V(I_{\hat{C}}(\hat{B})) \subset V(\hat{C}) \cap V(C) \subset V(I_{\tilde{C}}(\tilde{B})) \setminus \{\tilde{u}\}$, contradicting the minimality of $I_{\tilde{C}}(\tilde{B})$. <u>end of proof of claim 2</u>

It is an easy exercise to derive from Claim 2 that each odd circuit in (G,Σ) contains $\tilde{u}$. So (G,Σ) is almost bipartite. $\square$

Using the result just shown we can prove the main result of this section.

<u>Proof of Theorem 4.2.2</u>

Let G be a connected graph. Then we have: $\nu_T(G) = \tau_T(G)$ for every even subset T of V(G) if and only if

(*) *for each* $w \in \{-1,1\}^{E(G)}$ *such that* $\sum_{e \in E(C)} w_e \geqq 0$ *for each circuit* C *in* G *there exists a collection of edge disjoint coboundaries* $\delta(U_{\tilde{e}})$, $\tilde{e} \in \{e \in E(G) \mid w_e = -1\} (=: F_w)$, *such that* $\tilde{e} \in \delta(U_{\tilde{e}})$ *for each* $\tilde{e} \in F_w$.

(This equivalence follows from Lemmas 4.1.3 and 4.1.4.) Let G be a graph such that (G,E(G)) contains no odd-K_4 and no odd-prism, and such that Theorem 4.2.2 is correct for all graphs with fewer edges than G. We prove that (*) holds for G. So let $w \in \{-1,1\}^{E(G)}$ such that:

(**) $\sum_{e \in E(C)} w_e \geqq 0$ for each circuit C in G.

We consider the three cases of Theorem 4.2.3.

<u>Case I</u>: G *has a one node cutset,* {u} *say.*

It is not hard to see that now a packing with coboundaries, as meant in (*), is obtained by taking the union of such packings in each of the sides of the cutset {u}.

<u>Case II</u>: G *is two-connected, and has a strong 2-split.*

So G has two non-bipartite subgraphs G_1 and G_2 such that $V(G_1) \cup V(G_2) = V(G)$, $|V(G_1) \cap V(G_2)| = 2$ ($V(G_1) \cap V(G_2) = \{u,v\}$ say), $E(G_1) \cup E(G_2) = E(G)$, and $E(G_1) \cap E(G_2) = \emptyset$. For i = 1,2, let α_i be the length, with respect to w, of the shortest uv-path in G_i. By (**), $\alpha_1 + \alpha_2 \geqq 0$. Hence we may assume $\alpha_2 \geqq 0$.

Construct $\tilde{G}_1$ from G_1 by adding to G_1 a uv-path, P say, such that $|E(P)| = \alpha_2$ (If $\alpha_2 = 0$, identify u and v and call the new node u again.) Define $w^1 \in \{-1,1\}^{E(G_1)}$ by $w^1_e = 1$ if $e \in E(P)$ and $w^1_e = w_e$ if $e \in E(G_1)$. Now $(\tilde{G}_1, E(\tilde{G}))$ contains neither an odd-K_4, nor an odd-prism. (Indeed, there exist a uv-path Q in G_2 with $|E(Q)| \equiv \alpha_2 = |E(P)|$ (modulo 2).) Moreover $\tilde{G}_1$

contains no negatively weighted circuits with respect to w^1. So there exists a collection $\{\delta(U_e)|e\in F_{w'}\}$ of coboundaries in $\tilde{G}_1$, satisfying (*) with respect to w^1. We may assume $u\notin U_e$ for each $e\in F_{w^1}$. Define $Z :=$ $\{e\in F_{w^1}|\delta(U_e)\cap E(P) \neq \emptyset\}$, and $\rho := |Z|$.

Next we construct $\tilde{G}_2$ from G_2 by adding a uv-path Q to G_2 with $|E(Q)| = \rho$. (If $\rho = 0$, identify u and v, and call the new node u again.)

<u>Claim 1</u>: $(\tilde{G}_2, E(\tilde{G}_2))$ *contains neither an odd-*K_4 *nor an odd-prism.*
<u>Proof of Claim 1</u>: As G_1 is non-bipartite, and G is two-connected there exists in G_1 an even uv-path, as well as an odd uv-path.

<u>end of proof of claim 1</u>

Define $w^2\in\{-1,1\}^{E(\tilde{G}_2)}$ by $w_e^2 = -1$ if $e\in E(Q)$, and $w_e^2 = w_{e_2}$ if $e\in E(G_2)$. There are no negatively weighted circuits with respect to w^2 in $\tilde{G}_2$. (Note that $\rho \leqq \alpha_2$, and hence $-\rho + \alpha_2 \geqq 0$.) So as $\tilde{G}_2$ has fewer edges than G, there exists a collection $\{\delta(V_e)|e\in F_{w^2}\}$ of coboundaries in $\tilde{G}_2$ in the sense of (*) with respect to w^2. We may assume $u\notin\delta(V_e)$ for each $e\in F_{w^2}$.

In case $\rho \neq 0$ let π be some bijection from Z to E(Q). Now it is easy to see that

$$\{\delta(U_e)|e\in F_{w^1}\backslash Z\}\cup\{\delta(V_e)|e\in F_{w^2}\backslash E(Q)\}\cup\{\delta(U_e\cup V_{\pi(e)})|e\in Z\}$$

(or in case $\rho = 0$: $\{\delta(U_e)|e\in F_{w^1}\}\cup\{\delta(V_e)|e\in F_{w^2}\}$) is a collection of coboundaries in G, satisfying (*) with respect to w.

<u>Case III:</u> G *is almost bipartite.*

Let $u\in V(G)$ such that $G|(V(G)\backslash u)$ is bipartite, with bipartition U_1, U_2, say. Define $\tilde{G}$ as follows:

$V(\tilde{G}) = (V(G)\backslash\{u\})\cup\{u_1,u_2\}$;

$E(\tilde{G}) = (E(G)\backslash\delta(u))\cup\{vu_i|v\in U_i,\ vu\in E(G),\ i=1,2\}\cup\{u_1u_2\}$,

and $\tilde{w}_e\in\{-1,1\}^{E(\tilde{G})}$ by

$$\tilde{w}_e = \begin{cases} w_e & \text{if} \quad e \in E(G) \setminus \delta(u); \\ w_{vu} & \text{if} \quad e = vu_i;\ v \in V(G) \setminus \{u\};\ i = 1,2; \\ -1 & \text{if} \quad e = u_1u_2. \end{cases}$$

<u>Claim 2</u>: $\sum_{e \in E(C)} \tilde{w}_e \geqq 0$ *for all circuits* C *in* $\tilde{G}$.

<u>Proof of Claim 2</u>: Suppose to the contrary that $\sum_{e \in E(C)} \tilde{w}_e < 0$ for a circuit C in $\tilde{G}$. Obviously the edges in $E(C) \setminus \{u_1u_2\}$ give a circuit $\tilde{C}$ in G, hence $u_1u_2 \in C$. But this means that $\tilde{C}$ is odd in G, and so $\sum_{e \in E(C)} \tilde{w}_e = -1 + \sum_{e \in E(C)} w_e \geqq -1+1 = 0$. Contradiction. <u>end of proof of claim 2</u>

Since $\tilde{G}$ is bipartite, Theorem 4.1.1 yields the existence of a collection $\{\delta(U_e) | e \in F_{\tilde{w}}\}$ of coboundaries as meant in (*) with respect to $\tilde{w}$ in G. We may assume $u_1 \notin U_e$ ($e \in F_{\tilde{w}}$). But now $\{\delta(U_e) | e \in F_w \setminus \{u_1u_2\}\}$ is a desired collection of coboundaries with respect to w in G. □

<u>Remark</u>:

Case III in the proof above was derived independently by D. Wagner.

4.3. T-JOINS AND BINARY MATROIDS

A *graft* is a pair [G,T], where G is an undirected graph, and $T \subset V(G)$. Associated with a graft [G,T] we define the binary matroid $\mathcal{T}[G,T]$ as follows:
Let $x_T \in \mathbb{R}^{V(G)}$ be the characteristic vector of T as a subset of V(G), and let M_G be the node-edge incidence matrix of G. Then $\mathcal{T}[G,T]$ is the binary matroid represented over GF(2) by

$$\left[\begin{array}{c|c} M_G & x_T \end{array} \right].$$

The element of $\mathcal{T}[G,T]$, not in E(G), so corresponding with the last column of the above matrix will be denoted by t. So $E(\mathcal{T}[G,T]) = E(G) \cup \{t\}$.

CIRCUITS OF $\mathcal{T}[G,T]$

The circuits of $\mathcal{T}[G,T]$ are all sets of the forms:
- E(C), if C is a circuit in G;
- $E(F) \cup \{t\}$, if F is a minimal T-join in G.

RANK FUNCTION OF $\mathcal{T}[G,T]$

If $E' \subset E(G)$, then

$$r_{\mathcal{T}[G,T]}(E') = r_{\mathcal{M}(G)}(E'),$$

(4.3.1) $$r_{\mathcal{T}[G,T]}(E' \cup \{t\}) = r_{\mathcal{M}(G)}(E') \qquad \text{if } E' \text{ contains a T-join,}$$

$$r_{\mathcal{T}[G,T]}(E' \cup \{t\}) = r_{\mathcal{M}(G)}(E')+1 \qquad \text{if } E' \text{ contains no T-join.}$$

We define the following *reductions* of a graft [G,T]:
deletion $[G,T]\backslash e := [G\backslash e, T]$;
contraction $[G,T]/e := [G/e, T/e]$, where $T/e \subset V(G/e)$ is defined by:
$T/e := (T\backslash\{u,v\}) \cup v^*$ if $|\{u,v\} \cap T|$ is odd, and
$T/e := T\backslash\{u,v\}$ if $|\{u,v\} \cap T|$ is even.
Here v^* is the node of G/e in which e = uv is contracted.

Finally we also consider the deletion of isolated nodes not in T from G as a reduction of [G,T]. If $[\tilde{G},\tilde{T}]$ can be constructed from [G,T] by a series of reductions, we say that [G,T] *reduces to* $[\tilde{G},\tilde{T}]$. Obviously graft-reductions correspond to deletions and contractions in $\mathcal{T}[G,T]$.

MINORS OF $\mathcal{T}[G,T]$

$\mathcal{T}[G,T]/e = \mathcal{T}([G,T]/e)$, and

$\mathcal{T}[G,T]\backslash e = \mathcal{T}([G,T]\backslash e)$ for $e \in E(G)$.

Moreover $\mathcal{T}[G,T]\backslash t = \mathcal{M}(G)$ and $\mathcal{T}[G,T]/t$ is the binary matroid with circuits: all minimal T-joins, and all circuits in G containing no T-join.

Remark:

There is a similarity between grafts and signed graphs. Take an arbitrary T-join Σ in G. Then C is a circuit of $\mathcal{T}^*[G,T]$ if C is an minimal even coboundary or $t \in C$ and $C\backslash\{t\}$ is a minimal odd coboundary. Here odd (even) means containing an odd (even) number of edges from Σ. So $\mathcal{T}^*[G,T]$ is obtained from $\mathcal{M}^*(G)$ by signing similarly as $\mathcal{S}(G,\Sigma)$ is obtained from $\mathcal{M}(G)$. In particular if G is planar, with planar dual G^*, and T-join Σ, then $\mathcal{T}^*[G,T] = \mathcal{S}(G^*,\Sigma)$.

We define two special types of grafts: a *K_4-partition* and a *$K_{3,2}$-partition*. They are indicated in Figure 4.5. Circles stand for connected subgraphs, **odd** (**even**) indicates that the corresponding connected subgraph contains an odd (even) number of members in T, and lines stand for edges. In case each circle in Figure 4.5 contains exactly one point we speak of $\bar{K}_4$, $\bar{K}_{3,2}$ respectively. I.e., $\bar{K}_4 = [K_4, V(K_4)]$ and $\bar{K}_{3,2} = [K_{3,2}, V(K_{3,2})\backslash\{u\}]$, where u is one of the two nodes of degree three of the complete bipartite graph $K_{3,2}$. We say that a graft [G,T] *contains (or has) a K_4-partition* ($K_{3,2}$*-partition*) if each component of G contains an even number of points in T, and at least one component G_1 of G contains a subgraph $\tilde{G}_1$, with $V(G_1) = V(\tilde{G}_1)$, that is a K_4-partition ($K_{3,2}$-partition respectively).

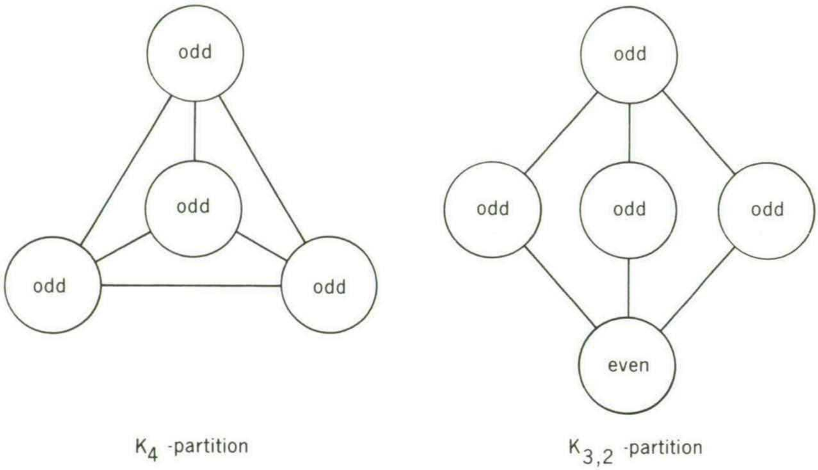


Figure 4.5

The following lemma is easy to prove:

Lemma 4.3.2

Let [G,T] *be a graft. Then the following are equivalent:*

(i) $\mathcal{T}$[G,T] *has an* F_7*-minor using* t;

(ii) [G,T] *reduces to* K_4;

(iii) [G,T] *contains a* K_4*-partition.*

Similarly, the following are equivalent:

(i) $\mathcal{T}$[G,T] *has an* F_7^**-minor using* t;

(ii) [G,T] *reduces to* $K_{3,2}$;

(iii) [G,T] *contains a* $K_{3,2}$*-partition.* □

Together with Tutte's characterization of regular matroids (cf. Theorem 1.4.4), this lemma yields

Lemma 4.3.3

Let [G,T] *be a graft. Then* $\mathcal{T}$[G,T] *is regular if and only if* [G,T] *contains no* K_4*-partition and no* $K_{3,2}$*-partition.* □

4.4. DECOMPOSITIONS

Grafts, and their associated binary matroids, where first introduced by Seymour [1980]. They play an important role in the proof of Seymour's decomposition theorem for regular matroids (cf. Theorem 3.2.1). In this section we shall interprete Seymour's result as well as the decomposition theorem of Truemper and Tseng [1986] (for binary matroids with no F_7^*-minor using a specific element), in terms of grafts. To this end we introduce the notion of *splits* for grafts.

Let $[G,T]$ be a graft, with $|T|$ even.

1-SPLIT:

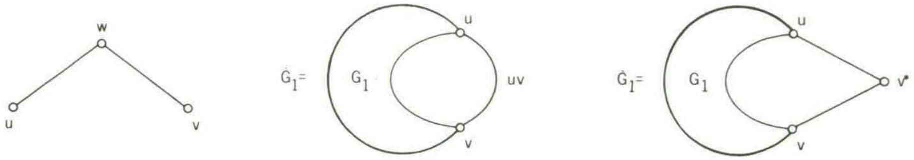
If G is disconnected, with component V_1, then $[G|V_1, T\cap V_1]$; $[G|(V(G)\backslash V_1), T\backslash V_1]$ is a 1-*split* of $[G,T]$.

If G is connected, and has a one-node cutset $\{u\}$, then $[G_1,T_1]$, $[G_2,T_2]$ is a 1-*split* of G, where G_1 and G_2 are the two sides of the cutset $\{u\}$, and T_1 is defined as $T\backslash V(G_2)$ in case $|T\cap V(G_2)|$ is even, and as $(T\backslash V(G_2))\cup\{u\}$ in case $|T\cap V(G_2)|$ is odd. T_2 is defined similarly. $[G_1,T_1]$ and $[G_2,T_2]$ are the *parts* of the 1-split.

2-SPLIT:

If G has a two-node cutset, $\{u,v\}$, say with sides G_1 and G_2, such that neither G_1 nor G_2 is equal to the graph in Figure 4.6(a) below, with $w\in T$, then $[\tilde{G}_1,\tilde{T}_1]$, $[\tilde{G}_2,\tilde{T}_2]$ is a 2-*split*, where $\tilde{G}_1,\tilde{T}_1$ is defined as follows: ($[\tilde{G}_2,\tilde{T}_2]$ is defined similarly.)

If $T\subset V(G_1)$, then $V(\tilde{G}_1) := V(G_1)$, $E(\tilde{G}_1) = E(G_1)\cup\{uv\}$; $\tilde{T}_1 := T$. (Figure 4.6(b)). If $T\backslash V(G_1) \neq \emptyset$, then $[\tilde{G}_1,\tilde{T}_1]$ is defined by $V(\tilde{G}_1) := V(G_1)\cup\{v^*\}$, (where v^* is a new node) and $E(\tilde{G}_1) = E(G_1)\cup\{uv^*,v^*v\}$ (Figure 4.6(c)). Moreover $\tilde{T}_1 : = (T\cap V(G_1))\cup\{v^*\}$ if $|T\backslash V(G_1|$ is odd and $\tilde{T}_1 :=$ $(T\cap V(G_1))\Delta\{u,v^*\}$ if $|T\backslash V(G_1)|$ is even. $[\tilde{G}_1,\tilde{T}_1]$ and $[\tilde{G}_2,\tilde{T}_2]$ are the *parts* of the 2-split. In case $T\backslash V(G_1)$ and $T\backslash V(G_2)$ both are nonempty we call the 2-split *strong*.



(a) (b) (c)

3-SPLIT:

If G has a three-node cutset, $\{u_1, u_2, u_3\}$ say, with two sides G_1 and G_2 such that: $TCV(G_1)$, $|E(G_2)| \geqq 4$, then $[\tilde{G}_1, T]$ is called a 3-*split*, where $\tilde{G}_1$ is defined by $V(\tilde{G}_1) = V(G_1) \cup \{v^*\}$ (where v^* is a new node); $E(\tilde{G}_1) = E(G_1) \cup \{u_1v^*, u_2v^*, u_3v^*\}$. $[\tilde{G}_1, T]$ is the *part* of the 3-split. (So a 3-split has one part only.)

The following lemma is easy to prove.

Lemma 4.4.1

Let $[G,T]$ *be a graft with a* k-*split* ($k \leqq 3$) *and no* ℓ-*split for any* $\ell <$ k. *Then* $[G,T]$ *has no* K_4-*partition and no* $K_{3,2}$-*partition if and only if each part of the* k-*split has no* K_4-*partition and no* $K_{3,2}$-*partition.*

Proof: Under the conditions given, each part of a split is a reduction of the original graft. This settles one side of the equivalence. The other side can be proved by case-checking. □

Now we state and prove a decomposition result for grafts with no K_4-partition and no $K_{3,2}$-partition.

Theorem 4.4.2

Let $[G,T]$ *be a graft containing no* K_4-*partition and no* $K_{3,2}$-*partition.*

Then one of the following holds:

(i) G *has a loop, or parallel edges;*

(ii) [G,T] *has a* 1-, 2-, *or* 3-*split;*

(iii) $|T|$ *is odd, or* $|T| \leq 2$*;*

(iv) G *is planar, with all members of* T *on one common face;*

(v) $G = K_{3,3}$, *and* $T = V(K_{3,3})$.

Proof: Let [G,T] be a graft with no K_4-partition and no $K_{3,2}$-partition. So, by Lemma 4.2.3, $\mathcal{T}[G,T]$ is regular. Hence we can apply Seymour's decomposition theorem (Theorem 3.2.1). We assume that [G,T] has neither a 1-, 2-, or 3-split, nor loops, nor parallel edges. Moreover we assume that $|T|$ is even. We consider four cases.

Case I: $\mathcal{T}[G,T]$ *is graphic.*

We prove that $|T| = 0$ or 2. Let $\mathcal{T}[G,T] \sim \mathcal{M}(\tilde{G})$ for some graph $\tilde{G}$. Let $e \in E(\tilde{G})$ correspond to t. If e is a loop, then t is a loop in $\mathcal{T}[G,T]$; so $T = \emptyset$. So suppose that $e = uv$ ($u \neq v$, $u,v \in V(\tilde{G})$). Observe that $\mathcal{M}(G) = \mathcal{T}[G,T]\backslash t \sim \mathcal{M}(\tilde{G}\backslash e)$. As G has no 1- or 2-split, each two-node cutset of G has one side equal to the graph of Figure 4.6(a). From Whitney's Theorem (if G_1 is 3-connected and $\mathcal{M}(G_1) \sim \mathcal{M}(G_2)$, then $G_1 \sim G_2$ (Whitney [1932])) it now follows that $G \sim \tilde{G}\backslash e$. So we may assume that $G = \tilde{G}\backslash e$. Take any uv-path P in $\tilde{G}\backslash e$ (= G). Then P together with e is a circuit in $\mathcal{M}(\tilde{G})$. So P together with t is a circuit in $\mathcal{T}[G,T]$. This implies $T = \{u,v\}$.

Case II: $\mathcal{T}[G,T]$ *is co-graphic.*

We prove that [G,T] satisfies (iv). Let $\tilde{G}$ be a graph such that $\mathcal{T}[G,T] \sim \mathcal{M}^*(\tilde{G})$. Let $e \in E(\tilde{G})$ be the edge corresponding to t. Then $\mathcal{M}(G) = \mathcal{T}[G,T]\backslash t \sim \mathcal{M}^*(\tilde{G})\backslash e = \mathcal{M}^*(\tilde{G}/e)$. So $\mathcal{M}(G)$ is graphic and co-graphic, and hence G is planar. If e is a loop in $\tilde{G}$, then t is a loop in $\mathcal{T}[G,T]$, and hence $T = \emptyset$ and (iv) holds. So suppose $e = uv$ with $u \neq v$, $u,v \in V(\tilde{G})$. As in Case I we may assume that the planar dual G^* of G satisfies $G^* = \tilde{G}/e$. Let $u^* \in V(G^*)$ be the node in which $\{u,v\}$ is contracted by the contraction $\tilde{G}/e$. Let F be the collection of edges in G, corresponding to $\delta(u)\backslash\{e\}$. As $\delta(u)$

is a circuit in $\mathcal{M}^*(\tilde{G})$, F is a T-join. But after the contraction of e, the edges in $\delta(u)\setminus\{e\}$ are in $\delta(u^*)\subset E(\tilde{G}/e)$. This means that the boundary C of the face in G corresponding to u^* in $G^* = \tilde{G}/e$ contains a T-join, namely F. That is, $T\subset V(C)$; so (iv) holds.

Case III: $\mathcal{T}[G,T] \sim \mathcal{R}_{10}$.

It is straightforward to verify that in this case $G = K_{3,3}$, $T = V(K_{3,3})$. (Note that $\mathcal{R}_{10}\setminus x \sim \mathcal{M}(K_{3,3})$ for each $x\in E(\mathcal{R}_{10})$.)

Case IV: $\mathcal{T}[G,T]$ *satisfies (1) of Theorem 3.2.1.*

We prove that (iii) or (iv) hold. So, let E_1, E_2 form a partition of E(G) such that

$$(*) \qquad r_{\mathcal{T}[G,T]}(E_1) + r_{\mathcal{T}[G,T]}(E_2\cup\{t\}) = r_{\mathcal{T}[G,T]}(E(G)\cup\{t\}) + k-1$$

with k=1,2 and $|E_1|$, $|E_2| + 1 \geqq k$, or k=3 and $|E_1|$, $|E_2| + 1 \geqq 6$.
From (*) and (4.3.1) we get:

$$(**) \qquad r_{\mathcal{M}(G)}(E_1) + r_{\mathcal{M}(G)}(E_2) = r_{\mathcal{M}(G)}(E(G)) + (k-\varepsilon)-1,$$

where $\varepsilon := 0$ if E_2 contains a T-join, and $\varepsilon := 1$ else.

Define $E_1^1,\ldots,E_1^s$, $E_2^1,\ldots,E_2^t$, and the auxilary graph H, as in the proof of Theorem 3.2.3. (Note that if $E_2 = \emptyset$, then $k = 1$ and $\varepsilon = 0$. So $T = \emptyset$, and hence (iii) holds.)

Claim 1: H *is a bipartite connected graph with no isthmuses. Moreover* $|E(H)| = s + t + k - \varepsilon - 2 = |V(H)| + k - \varepsilon - 2$.
Proof of Claim 1: The proof is similar to the proofs of Claims 1 and 2 in the proof of Theorem 3.2.3. end of proof of claim 1

Claim 2: $k = 3$, *and* $\varepsilon = 0$. H *is homeomorph is to the graph in Figure 4.7(b).*
Proof of Claim 2: If H is a circuit, then [G,T] would have a 2-split. Claim 1 now yields $k - \varepsilon - 2 \geq 1$. So $\varepsilon \leq k - 3$, i.e. $k = 3$, $\varepsilon = 0$. So

$|E(H)| = |V(H)| + 1$. Since H has no isthmuses, H is homeomorph to one of the graphs in Figure 4.7. If H is homeomorphic to the graph in Figure 4.7(a), then [G,T] must have a 2-split, a contradiction. So H is homeomorphic to the graph in Figure 4.7(b). <u>end of proof of claim 2</u>

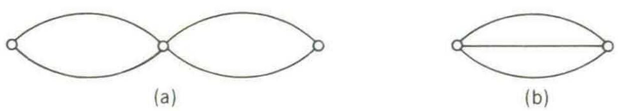


Figure 4.7

Hence G is of the form as in Figure 4.8 where $A, B \in \{E_1^1, \ldots E_1^s, E_2^1, \ldots, E_2^t\}$, and C_1, C_2 and C_3 are unions of elements of $\{E_1^1, \ldots, E_1^s, E_2^1, \ldots, E_2^t\} \setminus \{A, B\}$. Note that for i = 1, 2, 3 it is possible that $u_i = v_i$, implying $C_i = \emptyset$.

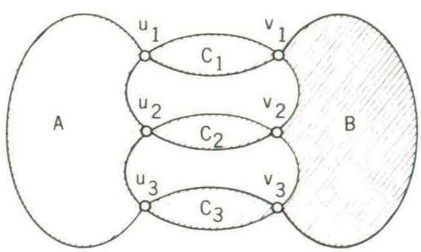


Figure 4.8

<u>Claim 3</u>: $C_i = \emptyset$, $C_i = \{u_iv_i\}$, *or* $C_i = \{u_iw_i, w_iv_i\}$ *for some* $w_i \in T$, *for* $i = 1, 2, 3$. *Moreover* $|C_1| + |C_2| + |C_3| \leq 5$.
<u>Proof of Claim 3</u>: The first part of the claim follows from the fact that [G,T] has no 2-split. If the second part would not be true, then $C_i = \{u_iw_i, w_iv_i\}$ for some $w_i \in T$ for each i = 1, 2, 3. But then [G,T] has a $K_{3,2}$-partition (T is even), a contradiction. <u>end of proof of claim 3</u>

<u>Claim 4</u>: $A \cup B = E_1$, *and* $C_1 \cup C_2 \cup C_3 = E_2$.
<u>Proof of Claim 4</u>: Since $|E_1| \geq 6$, E_1 cannot be contained in $C_1 \cup C_2 \cup C_3$. So we may assume $A = E_1^1$. Moreover, $|E_1^1| \leqq 3$, as [G,T] has no 3-split. The

edges in $C_1 \cup C_2 \cup C_3$ which are adjacent to u_1, u_2, or u_3 cannot be in E_1. (Since A is a component of E_2.) Now from Claim 3 and the fact that $|E_1| \geq 6$ if follows that $B = E_1^2$. Since $|E_2| \geq 5$, and $|C_1| + |C_2| + |C_3| \leq 5$ we have $C_1 \cup C_2 \cup C_3 = E_2$. <u>end of proof of claim 4</u>

<u>Claim 5</u>: G *is the graph in Figure 4.9. Moreover* $w_1, w_3 \in T$.
<u>Proof of Claim 5</u>: From the previous it follows that we only need to prove that $A = E_1^1$ and $B = E_1^2$ (cf. Figure 4.8) are triangles. If $|E_1^1|$ or $|E_1^2|$ is at least 4, then [G,T] has a 3-split. Since $|E_1| \geq 6$, this yields $|E_1^1| = |E_1^2| = 3$. If E_1^1 or E_1^2 is not a triangle then one easily finds a 1- or 2-split. <u>end of proof of claim 5</u>

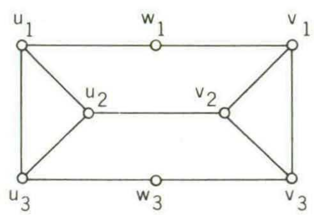


Figure 4.9

So $w_1, w_3 \in T$. If $u_2 \in T$, or $v_2 \in T$, then we would have a $K_{3,2}$-partition (as $|T|$ is even). Hence T lies on the outer face of the planar graph G, i.e. (iv) holds. This finishes the proof of Theorem 4.4.2. □

Also for graphs with no K_4-partition a decomposition result holds. It follows from Theorem 4.4.2 and the following result. (It also follows from Truemper [1987a: Theorem 2.1]. We give an elementary proof.)

<u>Theorem 4.4.3</u>
Let [G,T] *be a graft with no* K_4*-partition.*
Then one of the following holds:
(i) G *has parallel edges;*
(ii) [G,T] *has a 1-split or a strong 2-split;*
(iii) [G,T] *has no* $K_{3,2}$*-partition;*
(iv) $[G,T] \sim \bar{K}_{3,2}$.

<u>Proof</u>: Let $[G,T]$ be a graft with no K_4-partition, and not satisfying (i), (ii), or (iii). We shall prove that $[G,T] \sim \bar{K}_{3,2}$. First we define an *extended* $K_{3,2}$*-partition*, by Figure 4.10. The sets U^1, U^2, V_1^1, V_2^1, V_3^1, V_1^1, V_2^2, V_3^2 partition $V(G)$. The graphs induced by these sets are connected. For each $i = 1,2$ and $j = 1,2,3$: $|V_j^i \cap T|$ is odd, or $V_j^i = \emptyset$, and for each $j = 1,2,3$ we have that $V_j^1 \cup V_j^2 \neq \emptyset$.

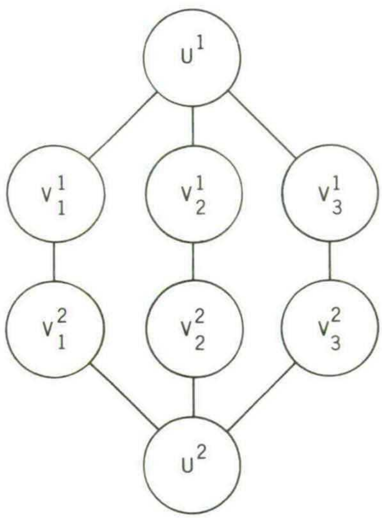


Figure 4.10

Since $[G,T]$ contains a $K_{3,2}$-partition, it contains an extended $K_{3,2}$-partition. Let U^1, U^2, etc. be an extended $K_{3,2}$-partition with $|U^1| + |U^2|$ minimal.

<u>Claim 1</u>: *Let* $i = 1,2$. *Then there exists a* $u_i \in U^i$ *and edges* $u^i v_1^i$, $u^i v_2^i$ *and* $u^i v_3^i$, *such that for each* $j = 1,2,3$: $v_j^i \in V_j^1 \cup V_j^2$, *and* $v_j^i \in V_j^1$ *if* $V_j^1 \neq \emptyset$.
<u>Proof of Claim 1</u>: Obviously we may assume that $i = 1$, and that $V_j^1 \neq \emptyset$ for $j = 1,2,3$. There exist a node $u \in U^1$ and three mutually openly disjoint paths P_1, P_2, and P_3 from u to $v_1 \in V_1^1$, $v_2 \in V_2^1$, and $v_3 \in V_3^1$ respectively, such that $V(P_j) \setminus \{v_j\} \subset U^1$ for $j = 1,2,3$. To prove the claim it suffices to prove

that each P_j $(j=1,2,3)$ is a single edge. By symmetry we may restrict ourselves to prove that P_1 is a single edge. Suppose this is not the case. Then the set, X say, of nodes $v \in U^1 \backslash (V(P_2) \cup V(P_3))$ for which there exists a vw-path P, with $w \in V_1^1$ and $V(P)\backslash\{w\} \subset U^1 \backslash (V(P_2) \cup V(P_3))$ is not empty. Define the sets $\tilde{V}_1^1$ and $\tilde{V}_1^2$ as follows (note that $V_1^1 \neq \emptyset$):

- if $|X \cap T|$ is odd, and $V_1^2 \neq \emptyset$, then $\tilde{V}_1^1 := X \cup V_1^1 \cup V_1^2$, and $\tilde{V}_1^2 := \emptyset$,
- if $|X \cap T|$ is odd, and $V_1^2 = \emptyset$, then $\tilde{V}_1^1 := X$, and $\tilde{V}_1^2 := V_1^1$,
- if $|X \cap T|$ is even, then $\tilde{V}_1^1 := X \cup V_1^1$ and $\tilde{V}_1^2 := V_1^2$.

The two sets $\tilde{V}_1^1$ and $\tilde{V}_1^2$ form, together with $\tilde{U}^1 := U^1 \backslash X$, $\tilde{U}^2 := U^2$, $\tilde{V}_j^i := V_j^i$ $(i=1,2;\ j=2,3)$, an extended $K_{3,2}$-partition. The fact that $|\tilde{U}_1| + |\tilde{U}_2| = |U_1| + |U_2| - |X| < |U_1| + |U_2|$ contradicts our assumption that $|U_1| + |U_2|$ is as small as possible. <u>end of proof of claim 1</u>

<u>Claim 2</u>: *Let* $i,j \in \{1,2,3\}$ *with* $i \neq j$. *Then there exists no edge from* $V_i^1 \cup V_i^2$ *to* $V_j^1 \cup V_j^2$.

<u>Proof of Claim 2</u>: If there was such an edge, one could easily find a K_4-partition in [G,T]. <u>end of proof of claim 2</u>

Define $\bar{U}^i := U^i \backslash \{u^i\}$ for $i = 1,2$, with u^1 and u^2 as in Claim 1. It is easy to derive from Claim 3 below that $[G,T] \sim \bar{K}_{3,2}$.

<u>Claim 3</u>: $\bar{U}^1 = \bar{U}^2 = \emptyset$, $|V_j^1 \cup V_j^2| = 1$ $(j=1,2,3)$ *and* $u^1u^2 \notin E(G)$.

<u>Proof of Claim 3</u>: First note that there exists no edge from $\bar{U}^1 \cup \bar{U}^2$ to $\bigcup_{j=1}^{3} \bigcup_{i=1}^{2} V_j^i$. Indeed, such edge would imply the existence of an extended $K_{3,2}$-partition with smaller $|U_1| + |U_2|$. So $\{u^1, u^2\}$ is a two-node cutset. From the fact that [G,T] has no strong 2-split Claim 3 easily follows.

<u>end of proof of claim 3</u>

□

<u>Lemma 4.4.4</u>

Let [G,T] *be a graft with no* $K_{3,2}$-*partition. If* [G,T] *has no 1- or 2-split, then* [G,T] *has no* K_4-*partition or* $[G,T] = \bar{K}_4$. □

<u>Remark</u>:

Obviously Lemma 4.4.1, Theorem 4.4.2, Theorem 4.4.3 and Lemma 4.4.4 yield polynomial-time algorithms for recognizing:

- grafts with no K_4-partition and no $K_{3,2}$-partition,
- grafts with no K_4-partition,
- grafts with no $K_{3,2}$-partition.

4.5. ORIENTATIONS

In Section 3.3 we characterized those graphs which have an orientation such that on each circuit the number of forwardly directed edges differs at most one from the number of backwardly directed edges. In this section we consider the question (posed by A. Frank): does there exist a "cut-version" of this result? To be precise: for which graphs G does there exist an orientation $\vec{A}$ of the edges such that for each inclusionwise minimal coboundary $\delta(U)$ the difference between the number of arcs in $\vec{A}$ entering U and the number of arcs in $\vec{A}$ leaving U, is at most one? An answer to this question is:

Theorem 4.5.1
Let G *be an undirected graph. Then the following two are equivalent:*
(i) $[G,\{v\in V(G)\,\big|\,|\delta(v)|\text{ is odd}\}]$ *contains neither a* K_4*-partition nor a* $K_{3,2}$*-partition;*
(ii) there exists an orientation $\vec{A}$ *of the edges in* G*, such that:* $\Big|\,|\{\overrightarrow{uv}\in\vec{A}\mid u\in U,\ v\notin U\}| - |\{\overrightarrow{uv}\in\vec{A}\mid u\notin U,\ v\in U\}|\,\Big| \leqq 1$ *for each* $U\subset V(G)$ *with the property that both* $G|U$ *and* $G|(V(G)\backslash U)$ *are connected.* □

This result is an immediate consequence of the following theorem.

Theorem 4.5.2
Let $[G,T]$ *be a graft with* G *connected and* $|T|$ *even. Then the following are equivalent:*
(i) $[G,T]$ *has no* K_4*-partition and no* $K_{3,2}$*-partition;*
(ii) there exists a partition T_1, T_2 *of* T *such that* $|T_1| = |T_2|$ *and each* T*-join is an edgedisjoint union of circuits and* $|T_1|$ *paths from* T_1 *to* T_2*;*
(iii) there exists a partition T_1, T_2 *of* T *such that for each* $U\subset V(G)$*, with* $G|U$ *and* $G|(V(G)\backslash U)$ *connected, we have* $\Big|\,|U\cap T_1| - |U\cap T_2|\,\Big| \leqq 1$*;*
(iv) for each T*-join* $F\subset E(G)$*, there exists an orientation* $\vec{A}$ *of the edges in* F *such that for each* $U\subset V(G)$*, with* $G|U$ *and* $G|(V(G)\backslash U)$ *connected, we have*

$$\Big|\,|\{\overrightarrow{uv}\in\vec{A}\mid u\in U,\ v\notin U\}| - |\{\overrightarrow{uv}\in\vec{A}\mid u\notin U,\ v\in U\}|\,\Big| \leqq 1.$$

Moreover, if a partition T_1, T_2 *of* T *satisfies (ii), then it satisfies (iii), and conversely.*

Proof:

(i) ⇒ (ii): If [G,T] has no K_4-partition and no $K_{3,2}$-partition then by Lemma 4.3.3 and Theorem 1.4.7 there exists a {0, ± 1}-matrix $[N,y] \equiv [M_G, x_T]$ (modulo 2) which represents $\mathcal{T}[G,T]$ over $\mathbb{R}$. We may assume that each column of N contains exactly one 1 and one -1 (cf. the Claim in the proof of Theorem 3.3.1). As the columns of M_G span x_T over GF(2), the columns of N span y. Hence y has as many 1's as -1's. Let $T_1 := \{u \in V(G) | y_u = 1\}$ and $T_2 := \{u \in V(G) | y_u = -1\}$. Then T_1 and T_2 partition T. Now let F be a T-join. Then there exists a {0, ± 1} vector $x \equiv x_F$ (modulo 2) such that Nx = y. (By Theorem 1.4.7, as $M_G x_F = x_T$.) It is easy to see that this means that F contains $|T_1|$ edgedisjoint paths from T_1 to T_2 with different endpoints. Now the fact that deleting these paths from F yields a cycle in G, proves (ii).

(ii) ⇒ (iii) and (iv): Let T_1 and T_2 be as in (ii). Take $U \subset V(G)$ with $G|U$ and $G|(V(G)\backslash U)$ connected. Then there exists a T-join $F \subset E(G)$ such that $|\delta(U) \cap F| \leqq 1$. Since T_1 and T_2 satisfy (ii) this means that $\big||U \cap T_1| - |U \cap T_2|\big| \leqq 1$. So (iii) follows.

To prove (iv), let F be a T-join in G. Let $P_1, \ldots, P_k$ ($k = |T_1|$) be paths from T_1 to T_2 and $C_1, \ldots, C_\ell$ be circuits in G such that $E(P_1), \ldots, E(P_k)$, $E(C_1), \ldots, E(C_\ell)$ partition F. Orient the edges on each path P_i ($i=1,\ldots,k$) such that each P_i becomes a directed path from T_1 to T_2. Orient the edges on each circuit C_i ($i = 1,\ldots,\ell$) such that C_i becomes a directed circuit. Let $\vec{A}$ be the orientation of F obtained in this way.

Take $U \subset V(G)$ with $G|U$ and $G|(V(G)\backslash U)$ connected. From $\big||U \cap T_1| - |U \cap T_2|\big| \leqq 1$, it easily follows that $\vec{A}$ satisfies the condition in (iv) with respect to U.

(iv) ⇒ (iii): Let F be a T-join and $\vec{A}$ be an orientation of F as meant in (iv). If $u \in V(G)$ is not a cutnode of G then $\big||\{\overrightarrow{uv} \in \vec{A} | v \in V(G)\}| - |\{\overrightarrow{vu} \in \vec{A} | v \in V(G)\}|\big| \leqq 1$. If u is a cutnode of G, the same inequality can be achieved by reversing, if necessary, all arcs of $\vec{A}$ at one side of the cutnode (by choosing the two sides appropriately). Now define $T_1 := \{u \in V(G) \big| |\{\overrightarrow{uv} \in \vec{A} | v \in V(G)\}| - |\{\overrightarrow{vu} \in \vec{A} | v \in V(G)\}| = 1\}$. Then $T_1 \subset T$. Let

$T_2 := T\backslash T_1$. Now it is easy to see that T_1 and T_2 satisfy the condition in (iii).
(iii) ⇒ (i): A partition of T as meant in (iii) is impossible for grafts with a K_4-partition or a $K_{3,2}$-partition, as is easily checked. □

In the following section we illustrate how Theorem 4.5.2 can be used to prove certain min-max relations for T-joins.

Remark:

Note that the decomposition result in Theorem 4.4.2 can be used not only to recognize grafts with no K_4-partition and no $K_{3,2}$-partition in polynomial-time, but also to find the partition T_1, T_2 of T as in Theorem 4.5.2 in polynomial-time. Indeed, if $|T| = 2$ the partition is obvious. In case G is planar with $T \subseteq V(C)$ for some face C of G then T_1 and T_2 are found as follows: Go along C, and put the nodes in T alternating in T_1 and in T_2. In case $[G,T] = [K_{3,3}, V(K_{3,3})]$ then T_1 and T_2 are the two colour classes of G. Finally if [G,T] has a 1-, 2-, or 3-split one finds T_1 and T_2 easily from the partition of $\tilde{T}$ into $\tilde{T}_1$ and $\tilde{T}_2$ in the parts of the split.

4.6. SHORTEST T-JOINS AND PACKING WITH T-JOINS

From Theorem 3.4.1 and Lemma 4.3.2 the following result follows.

Theorem 4.6.1

Let [G,T] *be a graft. Then the following are equivalent:*

(i) [G,T] *contains no* K_4*-partition;*

(ii) for each weight function $w \in \mathbb{Z}_+^{E(G)}$, *the minimum weight of a* T-*join is equal to the maximum cardinality of a* w-*packing with* T-*cuts.*

Similarly, the following are equivalent:

(i)' [G,T] *contains no* $K_{3,2}$*-partition;*

(ii)' for each weight function $w \in \mathbb{Z}_+^{E(G)}$, *the minimum weight of a* T-*cut is equal to the maximum cardinality of a* w-*packing with* T-*joins.* □

So in case [G,T] contains no K_4-partition and no $K_{3,2}$-partition, then both min-max relations in Theorem 4.6.1 hold. Below we shall see how this easily follows from the orientation Theorem 4.5.2. (See the remarks after Corollary 3.4.3.)

SHORTEST T-JOIN

Let $w \in \mathbb{Z}_+^{E(G)}$. The *shortest* T-*join problem* is:

(4.6.2) *Find a* T-*join* $F \subseteq E(G)$, *which minimizes* $\sum_{e \in F} w_e$.

The T-*cut packing problem* is

(4.6.3) *Find a maximum cardinality* w-*packing with* T-*cuts.*

Assume T_1 and T_2 forms a partition of T as is meant in Theorem 4.5.2. Replace all edges uv in G by two directed edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$. Call the set of arcs obtained in this way A. Consider the following primal-dual pair of linear programming problems. (If $\overrightarrow{uv} \in A$ then $w_{\overrightarrow{uv}} := w_{\overrightarrow{vu}} := w_{uv}$ $(uv \in E(G))$.)

(4.6.4) $\min \sum_{a\in A} w_a f_a$

s.t.

$$\sum_{a \text{ enters } u} f_a - \sum_{a \text{ leaves } u} f_a = \begin{cases} 1 & \text{if} \quad u\in T_1; \\ -1 & \text{if} \quad u\in T_2; \\ 0 & \text{if} \quad u\in V(G)\setminus T; \end{cases}$$

$$f_a \geqq 0 \quad \text{if} \quad a\in A.$$

(4.6.5) $\max \sum_{u\in T_1} \pi_u - \sum_{u\in T_2} \pi_u$

s.t. $\pi_v - \pi_u \leqq w_{\overrightarrow{uv}}$ if $\overrightarrow{uv}\in A$;

$\pi_v \in \mathbb{R}$ if $v\in V(G)$.

<u>Proposition 4.6.6</u>: *(4.6.2) and (4.6.4) are equivalent.*

<u>Proof</u>: Let F be a T-join. Since F is the disjoint union of $|T_1|$ paths from T_1 to T_2 and, possibly, some circuits, there exists a feasible solution f of (4.6.4) with $\sum_{a\in A} w_a f_a = \sum_{e\in F} w_e$.

Conversely, let $f\in \mathbb{Q}_+^A$ be an optimal solution of (4.6.3). As the constraint matrix of (4.6.4) is totally unimodular, we may assume that $f_a \in \mathbb{Z}$ for each $a\in A$. The set of arcs $F := \{uv\in E(G) \mid f_{\overrightarrow{uv}} + f_{\overrightarrow{vu}} \text{ is odd}\}$ is a T-join, with $\sum_{e\in F} w_e \leqq \sum_{a\in A} w_a f_a$.

Hence (4.6.2) is equivalent with (4.6.4). □

So we get the following (in)equalities between the optimal values of the above optimization problems:

max (4.6.3) $\leqq$ min (4.6.2) = min (4.6.4) = max (4.6.5).

So, in order to prove min (4.6.2) = max (4.6.3), it suffices to prove max (4.6.5) $\leqq$ max (4.6.3). Therefore, let $\pi\in \mathbb{Q}^{V(G)}$ be an optimal solution of (4.6.5). As the constraint matrix of (4.6.5) is totally unimodular we may assume that $\pi_u \in \mathbb{Z}$ $(u\in V(G))$. Define for each $\lambda\in\mathbb{R}$ with $\lambda_- :=$ $\min \{\pi_u \mid u\in V(G)\} \leqq \lambda \leqq \max \{\pi_u \mid u\in V(G)\} =: \lambda_+$ the set $V_\lambda := \{u\in V(G) \mid \pi_u \geqq \lambda\}$.

These sets V_λ satisfy the following two properties:

I: $$\sum_{\lambda=\lambda_-}^{\lambda_+} \left| |V_\lambda \cap T_1| - |V_\lambda \cap T_2| \right| \geqq \sum_{u\in T_1} \pi_u - \sum_{u\in T_2} \pi_u .$$

[Indeed, $$\sum_{\lambda=\lambda_-}^{\lambda_+} \left| |V_\lambda \cap T_1| - |V_\lambda \cap T_2| \right| \geqq \sum_{\lambda=\lambda_-}^{\lambda_+} (|V_\lambda \cap T_1| - |V_\lambda \cap T_2|) =$$

$$\sum_{\lambda=\lambda_-}^{\lambda_+} (\sum_{u\in T_1} |\{u\}\cap V_\lambda\}| - \sum_{u\in T_2} |\{u\}\cap V_\lambda|) =$$

$$\sum_{u\in T_1} \sum_{\lambda=\lambda_-}^{\lambda_+} |\{u\}\cap V_\lambda| - \sum_{u\in T_2} \sum_{\lambda=\lambda}^{\lambda_+} |\{u\}\cap V_\lambda| =$$

$$\sum_{u\in T_1} (\pi_u - \lambda_- + 1) - \sum_{u\in T_2} (\pi_u - \lambda_- + 1) = \sum_{u\in T_1} \pi_u - \sum_{u\in T_2} \pi_u .]$$

II: The collection $\delta(V_\lambda)(\lambda_- \leqq \lambda \leqq \lambda_+)$ is a w-packing with coboundaries.
[Straightforwardly.]

By applying the following proposition to each of the sets V_λ we find a w-packing with T-cuts with cardinality et least $\sum_{u\in T_1} \pi_u - \sum_{u\in T_2} \pi_u$. This proves max (4.6.3) $\geqq$ max (4.6.5).

<u>Proposition 4.6.7</u>: *Let* $U \subset V(G)$. *Then* $\delta(U)$ *contains at least* $\left| |U\cap T_1| - |U\cap T_2| \right|$ *disjoint T-cuts.*

<u>Proof</u>: First assume $G|U$ to be connected. Let $V_1,\ldots,V_k$ be the node sets of the components of $G|(V(G)\setminus U)$, with $|V_i\cap T|$ odd for $i=1,\ldots,\ell$, and even for $i=\ell+1,\ldots,k$. As $\delta(V_1),\ldots,\delta(V_k)$ partition $\delta(U)$, we only need to prove that $\ell \geqq \left| |U\cap T_1| - |U\cap T_2| \right|$:

$$\left| |U\cap T_1| - |U\cap T_2| \right| = \left| |(V(G)\setminus U)\cap T_1| - |(V(G)\setminus U)\cap T_2| \right| \leqq$$

$$\sum_{i=1}^{k} \left| |V_i\cap T_1| - |V_i\cap T_2| \right| = \ell,$$

where the last equality follows since the pair T_1, T_2 satisfies Theorem 4.5.2(ii) and $G|V_i$ and $G|(V(G)\setminus V_i)$ are connected for $i=1,\ldots,k$.

Next consider the case that $G|U$ is disconnected. Let $U_1,\ldots,U_k$ be the node sets of the components of $G|U$. Above we proved the proposition for connected induced subgraphs of G. Applying this to U_i for $i=1,\ldots,k$ we get that $\delta(U_i)$ contains $\big||U_i\cap T_1| - |U_i\cap T_2|\big|$ disjoint T-cuts for $i=1,\ldots,k$. Now the proposition follows since $\delta(U_1),\ldots,\delta(U_k)$ partition $\delta(U)$, and

$$\big||U\cap T_1| - |U\cap T_2|\big| = \sum_{i=1}^{k}\big||U_i\cap T_1| - |U_i\cap T_2|\big|. \qquad \square$$

Conclusion:

We showed that the minimum in (4.6.2) equals the maximum in (4.6.3) for grafts with no K_4-partition and no $K_{3,2}$-partition. Implicitly we showed that (4.6.2) and (4.6.3) can be solved by solving a circulation problem ((4.6.4)) and its dual ((4.6.5)), as soon as T_1 and T_2 are known. It is interesting to note that in case T_1 and T_2 are known, Edmonds' algorithm in Section 4.1 can be simplified in the sense that only a minimum weight perfect matching in the complete bipartite graph with colour classes T_1 and T_2 has to be found.

As mentioned the min-max relation min (4.6.2) = max (4.6.3) holds in grafts with no K_4-partition. This follows from the just proved case (no K_4-partition and no $K_{3,2}$-partition) by using Theorem 4.4.3. (Compare with Cases I and II in the proof of Theorem 4.2.2.) See also Truemper [1987a] for deriving min-max relations for matroids with no F_7^*-minor using a specific element from such min-max relations for regular matroids, using decomposition.

PACKING T-JOINS

Let $w\in \mathbb{Z}_+^{E(G)}$. The T-*join packing problem* is:

(4.6.8) *Find a maximum cardinality* w-*packing with* T-*joins.*

The *shortest* T-*cut problem* is:

(4.6.9) *Find a* T-*cut* $\delta(U)\subset E(G)$, *which minimizes* $\sum_{e\in\delta(U)} w_e$.

Assume T_1 and T_2 from a partition of T as is meant in Theorem 4.5.2. Replace all edges uv in G by two directed edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$. Denote the set of arcs obtained in this way by A. Consider the following primal-dual pair of linear programming problems (If $\overrightarrow{uv}\in A$ then $w_{\overrightarrow{uv}} := w_{\overrightarrow{vu}} := w_{uv}$ $(uv\in E(G))$.

(4.6.10) max k

s.t.

$$\sum_{a \text{ enters } u} f_a - \sum_{a \text{ leaves } u} f_a = \begin{cases} k & \text{if} \quad u\in T_1; \\ -k & \text{if} \quad u\in T_2; \\ 0 & \text{if} \quad u\in V(G)\setminus T; \end{cases}$$

$$f_{\overrightarrow{uv}} + f_{\overrightarrow{vu}} \leqq w_{uv} \quad \text{if} \quad uv\in E(G);$$

$$f_a \geqq 0 \quad \text{if} \quad a\in A.$$

(4.6.11) $\min \sum_{e\in E(G)} w_e \ell_e$

$$\text{s.t. } \pi_v - \pi_u + \ell_{uv} \leqq 0 \quad \text{if} \quad \overrightarrow{uv}\in A;$$

$$\ell_e \geqq 0 \quad \text{if} \quad e\in E(G);$$

$$\pi_u \in \mathbb{R} \text{ if } u\in V(G);$$

$$\sum_{u\in T_1} \pi_u - \sum_{u\in T_2} \pi_u = 1.$$

In order to prove that (4.6.10) is equivalent with (4.6.8) we derive the following propositions.

<u>Proposition 4.6.12</u>: *Problem (4.6.10) has an integral optimal solution.*

<u>Proof</u>: Let $k^*\in\mathbb{Q}$, $f^*\in\mathbb{Q}^A$ be an optimal solution, that is not a convex combination of other optimal solutions. Obviously, it suffices to show that $k^*\in\mathbb{Z}$. (Observe the construction in Figure 3.11.)

Let $E' := \{uv\in E(G) \mid 0 < f_{\overrightarrow{uv}} + f_{\overrightarrow{vu}} < w_{uv}\}$, and let $V_1,\dots,V_\ell$ be the components of the subgraph G' of G with $V(G') := V(G)$, $E(G') = E'$. If E' contains a T-join, then one easily shows that k^*, f^* is not an optimal solution for (4.6.10). (See the first part of the proof of Proposition 4.6.6 to find a $\tilde{k} > 0$ and an $\tilde{f}$ such that $k^* + \tilde{k}$, $f^* + \tilde{f}$ is feasible for (4.6.10).) Hence $|V_i \cap T|$ is odd for some $i=1,\dots,\ell$. Therefore $G|(V(G)\setminus V_i)$

has at least one component with node set W (say) such that $|W \cap T|$ is odd. This set W satisfies the following properties:

- $f^*_a \in \mathbb{Z}$ if $a \in \delta(W)$ (as $\delta(W) \subseteq \delta(V_i)$);

- $|W \cap T_1| - |W \cap T_2| = \pm 1$ (as $G|W$ and $G|(V(G)\backslash W)$ are connected).

Combining these two properties with the feasibility of k^*, f^* for (4.6.10) we get: $\pm k^* = |W \cap T_1| k^* - |W \cap T_2| k^* = \sum_{a \text{ leaves } W} f^*_a - \sum_{a \text{ enters } W} f^*_a \in \mathbb{Z}$. This contradicts our assumption that $k^* \notin \mathbb{Z}$. □

Proposition 4.6.13: *Let* $k \in \mathbb{Z}_+$, $f \in \mathbb{Z}_+^A$ *be a feasible solution, with* $k \geqq 1$. *Then there exists a solution* $\tilde{k} \in \mathbb{Z}_+$, $\tilde{f} \in \mathbb{Z}_+^A$ *with* $\tilde{k}=1$, *such that for each* $a \in A$: $\tilde{f}_a \leqq f_a$.

Proof: Define the following capacitated digraph D:

$V(D) := V(G) \cup \{s,t\}$. (s and t are two new nodes);

$A(D') := A \cup \{\overrightarrow{su} | u \in T_1\} \cup \{\overrightarrow{ut} | u \in T_2\}$;

$c_a := f_a (a \in A)$; $c_{\overrightarrow{su}} := 1$ $(u \in T_1)$; $c_{\overrightarrow{ut}} := 1$ $(u \in T_2)$.

The statement in the proposition is equivalent with the existence of a flow from s to t in D with value $|T_1|$ and satisfying the capacities. So suppose such flow does not exist. Then from the max-flow min-cut Theorem of Ford and Fulkerson ([1956]) there exists a set $U \subset V(G)$ such that

$$\sum_{\substack{a \in A(D') \\ a \text{ leaves } U \cup \{s\}}} c_a < |T_2|.$$

Hence
$$\sum_{\substack{a \in A \\ a \text{ leaves } U}} f_a + |T_2 \backslash U| + |T_1 \cap U| < |T_2|.$$

As k and f form a feasible solution to (4.6.10), we have

$$\sum_{\substack{a \in A \\ a \text{ leaves } U}} f_a \geqq \max\{0,\ k|T_2 \cap U| - k|T_1 \cap U|\}.$$

Combining the last two inequalities we get

$$\max\{0,\ k|T_2\cap U| - k|T_1\cap U|\} < |T_2\cup U| - |T_1\cup U|,$$

which contradicts $k \geqq 1$. □

Corollary 4.6.14: *(4.6.8) and (4.6.10) are equivalent.*

Proof: The fact that each w-packing with $\tilde{k}$ T-joins yields a feasible solution of (4.6.10) of value $\tilde{k}$ is obvious. Conversely, let $f^*\in\mathbb{Q}_+^A$, $k^*\in\mathbb{Q}_+$ be an optimal solution of (4.6.10).

From Proposition 4.6.12 it follows that we may assume that f^* and k^* are integer valued. Now Proposition 4.6.13 yields the existence of $f^i\in\mathbb{Z}_+^A$ $(i=1,\ldots,k^*)$ such that $\sum_{i=1}^{k} f^i \leqq f^*$ and such that f^i together with $k=1$ forms a feasible solution to (4.6.10) for each $i=1,\ldots,k^*$. Hence the collection $F_i := \{uv\in E(G)\,|\, f^i_{\overrightarrow{uv}} + f^i_{\overrightarrow{vu}} \text{ is odd}\}$ $(i=1,\ldots,k^*)$ forms a w-packing of T-cuts in G. □

To prove that max (4.6.8) = min (4.6.9) we now only need to prove now the following proposition.

Proposition 4.6.15: *(4.6.9) is equivalent with (4.6.11).*

Proof: First, let $\delta(U)$ be a minimum weight T-cut. By Proposition 4.6.7 we may assume that $|U\cap T_1| - |U\cap T_2| = 1$. Define: $\pi_u := 1$ if $u\in U$; $\pi_u := 0$ if $u\in V(G)\setminus U$; $\ell_e := 1$ if $e\in\delta(U)$ and $\ell_e = 0$ if $e\in E(G)\setminus(\delta(U))$. Then π and ℓ form a feasible solution of 4.6.11. Moreover $\sum_{e\in\delta(U)} w_e = \sum_{e\in E(G)} w_e\ell_e$.

Conversely, let $\pi\in\mathbb{Q}^{V(G)}$, $\ell\in\mathbb{Q}^{E(G)}$ be an optimal solution of 4.6.11. By Proposition 4.6.13 and Corollary 1.2.19 we may assume ℓ to be integer valued. Since $\sum_{u\in T_1}\pi_u - \sum_{u\in T_2}\pi_u = 1$, there exists a $\lambda\in\mathbb{Q}$ such that $\tilde{V} := \{u|\pi_u = \lambda\}$ satisfies $|\tilde{V}\cap T_1| \neq |\tilde{V}\cap T_2|$. Obviously $\ell_e \geqq 1$ for each $e\in\delta(\tilde{V})$. By Proposition 4.6.7, there exists a T-cut $\delta(U)\subset\delta(\tilde{V})$. This T-cut $\delta(U)$ satisfies: $\sum_{e\in\delta(u)} w_e \leqq \sum_{e\in\delta(\tilde{V})} w_e \leqq \sum_{e\in E(G)} w_e\ell_e$. □

Conclusion:

From Corollary 4.6.14, Proposition 4.6.15 and linear programming duality (for (4.6.10) and (4.6.11)) we see that for grafts with no K_4-partition and no $K_{3,2}$-partition the maximum in (4.6.8) is equal to the minimum in (4.6.9).

To extend this result to grafts with no $K_{3,2}$-partition (see Theorem 4.6.1) one can use Lemma 4.4.4. (Cf. Truemper [1987a] for the general way to use decomposition to derive min-max relations for binary matroids with no F_7-minor containing some fixed element from the fact that these min-max relations hold for regular matroids.)

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SUBJECT INDEX

SAMENVATTING

GRAFEN EN POLYEDERS
Binaire Ruimten en Sneden

Beschouw het volgende combinatorische optimaliseringprobleem. Zij G een graaf met puntenverzameling V en kantenverzameling E. Een *stabiele verzameling* in G is een collectie $S \subset V$ zo dat $u,v \in S \Rightarrow uv \notin E$. Het *gewogen stabiele verzameling probleem* in G is:

(1) Gegeven $w \in \mathbb{Z}^V$. Vind een stabiele verzameling S met $\sum_{u \in S} w_u$ maximaal.

Een veel gebruikte aanpak voor een probleem als (1), is het probleem te formuleren als een geheeltallig lineair programmeringsprobleem:

(2) $\max\{w^T x \mid x \in P(G) \cap \mathbb{Z}^V\}$

waarbij P(G) de collectie vectoren $x \in \mathbb{R}^V$ is die voldoen aan

(3)
$$x_u \geqq 0 \quad (u \in V);$$
$$x_u + x_v \leqq 1 \quad (uv \in E).$$

Daar lineaire programmering in het algemeen makkelijker is dan geheeltallige programmering, schrijven we (2) als

(4) $\max\{w^T x \mid x \in P(G)_I\}$

met $P(G)_I :=$ convex omhulsel $(P(G)_I \cap \mathbb{Z}^V)$. Aangezien $P(G)_I$ een polyeder is, d.w.z. $P(G)_I = \{x \in \mathbb{R}^V \mid Mx \leqq d\}$ voor een zeker stelsel lineaire ongelijkheden $Mx \leqq d$, is (4) een lineair programmeringsprobleem. Om hierop lineaire programmerings technieken toe te kunnen passen is de existentie van $Mx \leqq d$ echter niet voldoende, we moeten zo'n stelsel (min of meer) expliciet kennen. Een van de manieren om dit te bereiken is door het toevoegen van sneden. Een *snede* voor P(G) is een ongelijkheid

$$(5)\qquad c^Tx \leqq \lfloor\delta\rfloor \quad \text{met } c\in\mathbb{Z}^V \quad \text{en } \delta \geqq \max\{c^Tx \mid x\in P(G)\}.$$

Een voorbeeld voor zo'n snede voor P(G) is

$$(6)\qquad \sum_{i=1}^{2k+1} x_{u_i} \leqq k \quad \text{waarbij } u_1u_2,\ u_2u_3,\ldots,\ u_{2k+1}u_1 \in E.$$

[Want $\sum_{i=1}^{2k+1} x_{u_i} = \frac{1}{2}(x_{u_1}+x_{u_2}) + \frac{1}{2}(x_{u_2}+x_{u_3}) + \ldots + \frac{1}{2}(x_{u_{2k+1}}+x_{u_1}) \leqq$ $\frac{1}{2}(2k+1) = k + \frac{1}{2}$ voor $x\in P(G)$. (6) heet een *oneven circuit ongelijkheid.*]

We definiëren $P(G)' := \{x\in\mathbb{R}^V \mid x$ voldoet aan alle sneden (5) voor $P(G)\}$. Vanzelfsprekend geldt $P(G)\supset P(G)'\supset P(G)_I$, dus is (2) equivalent met

$$(7)\qquad \max\{w^Tx \mid x\in P(G)'\cap\mathbb{Z}^V\}.$$

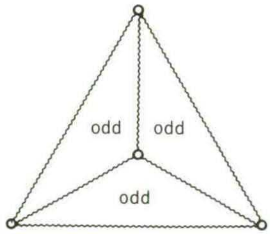
Alhoewel er oneindig veel sneden voor P(G) zijn, volstaat een eindige selectie sneden om P(G)' te beschrijven. Met name geldt dat P(G)' bestaat uit alle x die voldoen aan (3) en aan alle oneven circuit ongelijkheden (6).

Als $P(G)' \neq P(G)_I$ dan volgt een nieuwe ronde sneden, nu voor P(G)'. Dit levert $P(G)^{(2)} := P(G)''$. Een derde ronde levert $P(G)^{(3)}$, etc. Deze procedure wordt voortgezet tot na, zeg k, ronden $P(G)^{(k)} = P(G)_I$. Chvátal [1973] and Schrijver [1980] bewezen dat, voor elk polyeder P geldt dat deze procedure na een eindig aantal ronden succesvol is. Het benodigde aantal ronden heet de Chvátal-rang van het polyeder. Er zijn aanwijzigingen dat de moeilijkheid van een geheeltallig programmerings probleem als (2) toeneemt naarmate de Chvátal-rang van het polyeder (P(G) in (2)) toeneemt. Het eenvoudigste geval is dat de Chvátal-rang 0 is, dat wil zeggen $P(G) = P(G)_I$: er is geen ronde sneden nodig. (In ons concrete geval geldt dit dan en slechts dan als G bipartiet is.) In paragraaf 2.3 van Hoofdstuk 2, dat geheel aan deze snedenmethode gewijd is, beschouwen we polyeders met Chvátal-rang 1. In deze paragraaf bewijzen we de centrale stelling in dit proefschrift (Stelling 2.3.3). Hierin wordt voor een zekere klasse matrices A bewezen dat $\{x \mid Ax \leqq b\}' = \{x \mid Ax \leqq b\}_I$ voor elke geheeltallige vector b.

Voor P(G) levert deze stelling:

(8) Zij G een graaf zonder oneven-K_4. Dan is $P(G)' = P(G)_I$.

Een oneven-K_4 is een graaf als in onderstaande figuur. De kronkellijntjes zijn paden, en het woord **odd** in een gebied geeft aan dat de rand van het betreffende gebied een oneven circuit is.



In feite is (8) niet alleen een gevolg van Stelling 2.3.3, maar vormt min of meer ook het generieke geval daarin. Vandaar dat de klasse grafen zonder oneven-K_4 nader bestudeerd wordt in Hoofdstuk 3. Dit ondermeer teneinde een polynomiale algoritme te verkrijgen om na te gaan of een gegeven matrix A tot de klasse beschreven in Stelling 2.3.3 behoort. Hiervoor wordt gebruik gemaakt van de theorie van binaire matroiden (= lineaire ruimten over GF(2)), in het bijzonder van stellingen van Seymour [1980] en van Truemper en Tseng [1986]. Een andere stelling over binaire matroiden, Tutte's karakterisering van reguliere matroiden (= binaire ruimten representeerbaar in een euclidische ruimte, Tutte [1958]), wordt gebruikt om de klasse van grafen G te karakteriseren met de volgende eigenschap:

(9) Het is mogelijk de kanten van G zodanig te vervangen door gerichte kanten dat voor ieder circuit het aantal kanten gericht in de ene richting hooguit 1 verschilt van het aantal kanten gericht in de tegenovergestelde richting.

Dit resultaat leidt onder andere tot een generalisatie van (8).

De resultaten in Hoofdstuk 3 worden gegeven in de context van gesigneerde grafen, waarbij onderscheid gemaakt wordt tussen even en oneven kanten. De theorie van binaire matroiden toegepast in Hoofdstuk 3 op gesigneerde grafen, wordt in Hoofdstuk 4 toegepast op "grafts", dat wil zeggen grafen met een onderscheid tussen even en oneven punten. De resultaten in Hoofdstuk 4 (met uitzondering van stelling 4.2.2) zijn in zekere zin duaal aan die in Hoofdstuk 3.